Spin-density operator for the interacting two-spins systems

1985 J. Phys. A: Math. Gen. 18 671

(http://iopscience.iop.org/0305-4470/18/4/016)

J. Phys. A: Math. Gen. 18 (1985) 671-704. Printed in Great Britain

Spin-density operator for the interacting two-spins systems

Zbigniew Oziewicz†

Institute of Theoretical Physics, University of Wrocław, Cybulskiego 36, 50-205 Wrocław, Poland

Received 24 July 1984

Abstract. For the system of the two interacting *arbitrary* spins on the K-shell we derive the multipole expansions of the spin-density operator in terms of the total spin operator as well as in terms of the individual spin operators. The density operator is described by means of the truly independent set of parameters: the populations, degrees of orientation and the directions of orientation. For the case of the cylindrical symmetry, the degrees are expressed by means of the moments. The paper contains many examples which are important for applications in muon and nuclear physics. The results should serve for the phenomenological analysis of the decays of the muonic atoms and other nuclear reactions. The detailed discussion of the simplest model of the depolarisation due to the spin-spin hyperfine interaction is also included.

1. Introduction

There are several treatments of the parametrisation of the spin-density operator for the case of the *one* arbitrary spin. The multipole expansion of the spin-density operator for this case, including the discussion of the notions of the orientation and of the polarisation, is rather exhaustively presented by Werle (1966), Csonka *et al* (1966) and Steiger and Fritz (1967) which contains the accurate treatment of several particular problems. For early discussions of this subject one should refer to the important paper by Tolhoek and Cox (1953).

However, there is no systematic consideration of the interacting (correlated) system of *two arbitrary spins*. Here we consider mostly the spin-density operator diagonal in the total spin. The results should be valuable for the phenomenological analysis of some reactions in atomic and nuclear physics involving systems with non-zero spins. In particular we have in mind the nuclear muon capture process by oriented targets (Bukhvostov and Popov 1964, Bukhvostov *et al* 1972, Hambro and Mukhopadhyay 1975, Mukhopadhyay 1977) as well as the theory of exotic atoms. Some of our results can also find applications in the complicated theory of the atomic depolarisation of muons, as developed by Shmushkevich (1959) and Djrbashyan (1959) but mostly by Bukhvostov (1966, 1969).

Section 2 collects together all the necessary definitions and formulae related to the tensor operators which will be used subsequently. The theory of tensor operators is presented in such excellent and well known works, as e.g. Edmonds (1957), Fano and Racah (1959), Varschalovich *et al* (1975), Jucys and Bandzaitis (1977). We refer also to the recent paper by Klimyk (1983) where the tensor operators are defined in a basis-free manner. We will give some comments related to Klimyk's paper. The

0305-4470/85/040671+34\$02.25

polarisation operators of *arbitrary* rank are built up entirely from the total spin, tensor operator $\mathscr{F}$ (i.e. from the Lie algebra representation) and projectors.

In §§ 4 and 5 we derive the multipole expansion of the spin-density operator $\rho \equiv \rho(j_1 \otimes j_2) \in \mathrm{End}([j_1] \otimes [j_2])$ where $[j]$ denotes the $(2j+1)$-dimensional irreducible representation of the rotation group SO(3). If the spin-density operator is diagonal in the total spin then the multipole expansion can be given for *arbitrary* spin j_1 and j_2, in terms of the total spin operator $\mathscr{F} = \boldsymbol{j}_1 + \boldsymbol{j}_2$. Then the parametrisation of the spin-density operator is related to the mean values of the polynomials in $\mathscr{F}$, which is important for the physical interpretation of the Fano polarisation tensors (coefficients) and the device of their measurements. Such a spin-density operator is finally completely described by the three kinds of *truly independent* sets of the parameters: populations $\{p_F\}$, non-negative degrees of the orientation $\{\lambda_F^L\}$ and the directions of orientation {unit vectors: e_F^L}, which will be denoted by Pop, Deg and Dir respectively. The set of all possible values of the channel spin will be denoted by $\Delta(j_1, j_2)$ or abbreviated to Δ: $\Delta(j_1, j_2) \equiv \{|j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2\}$. Then we define

$$\mathrm{Pop}(j_1, j_2) \equiv \left\{ p_F \in [0, 1], \forall F \in \Delta(j_1, j_2), \sum_F p_F = 1 \right\} \tag{1}$$

$$\mathrm{Deg}(j_1, j_2) \equiv \left\{ \lambda_F^L \in [0, 1], \forall F \in \Delta(j_1, j_2), \forall L \in \{1, 2, \ldots, 2F\}, \sum_{F \in \Delta} \sum_{L=1}^{2F} (\lambda_F^L)^2 \leqslant 1 \right\}, \tag{2}$$

$$\mathrm{Dir}(j_1, j_2) \equiv \left\{ \sum_{F \in \Delta} \oplus \left(\sum_{L=1}^{2F} \oplus S^{2L} \right) \right\}. \tag{3}$$

Evidently dim $\mathrm{Pop}(j_1, j_2) = 2\min(j_1, j_2)$ in the sense that $\mathrm{Pop}(j_1, j_2)$ is the closed subset of $\mathbb{R}^{2\min(j_1, j_2)}$. In the same sense dim $\mathrm{Deg}(j_1, j_2) = 2 \Sigma F = 4j_1 j_2 + 2\max(j_1, j_2)$. In fact Deg is the closed positive segment of the closed ball. Dir is the Cartesian product of the even-dimensional spheres, i.e. this is a manifold. One can easily obtain dim $\mathrm{Dir} = 2 \Sigma F(2F+1) = N(N+1)$ where $N = \dim \mathrm{Pop} + \dim \mathrm{Deg}$. For the case $j_1 j_2 = 0$ the above is reduced to the well known situation (see e.g. Werle 1966) when there is no Pop and dim $\mathrm{Deg} = 2(j_1 + j_2) \equiv 2j$.

The manifold $\mathrm{Dir}(j_1, j_2)$ for the particular cases is explicitly

$$\mathrm{Dir}(0, j) = S^2 \times S^4 \times \ldots \times S^{4j}, \qquad \mathrm{Dir}(\tfrac{1}{2}, \tfrac{1}{2}) = S^2 \times S^4$$

$$\mathrm{Dir}(\tfrac{1}{2}, 1) = S^2 \times S^2 \times S^4 \times S^6, \qquad \text{etc.}$$

We show how these parameters (1)–(3) are related to the mean values of the operators which are built up from the total spin operator $\mathscr{F}$ and projectors.

The most important conclusion (see §§ 4 and 5) is that the natural rather than the popular presentation of the spin-density operator in the factorised form:

$$\rho(j_1 \otimes j_2) = \sum_{F \in \Delta} p_F \rho_F, \tag{4}$$

with the *population independent* operators ρ_F: $\mathrm{Tr}\, \rho_F = 1$, is in fact incorrect. In (4) the parametrisation of ρ_F does depend on the populations in some simple way; this follows from the calculations of $\mathrm{Tr}\, \rho^2 \leqslant 1$. The factorised form (4) has been employed in all previous publications, see e.g. Bukhvostov and Popov (1967, 1970), Bukhvostov *et al* (1972), Ciechanowicz and Mukhopadhyay (1978), Ciechanowicz and Oziewicz (1984). The factorised form (4), which suggests incorrectly the population independence of the ρ_F operators, does *not* of course invalidate any of the previously published results.

However, our multipole expansion (see (38) below), as opposed to the factorised form (4), is more *convenient.*

We devote particular attention to the cylindrical symmetry in which case $\mathrm{Dir}(j_1 j_2)$ is reduced to the discrete set. In § 6 we will derive the general and still very simple expression (68) of the degrees of the orientation $\{\lambda^L_F\}$ in terms of the so-called moments, i.e. the powers of the operator $m \equiv \mathcal{F}(s)$, where s is the unit vector of the cylindrical symmetry.

In § 7 the multipole expansion of the spin-density operator in terms of the individual spin operators

$$\jmath_1 = \mathcal{S}_{j_1} \otimes \mathrm{i}d \qquad \text{and} \qquad \jmath_2 = \mathrm{i}d \otimes \mathcal{S}_{j_2} \tag{5}$$

in the space $\mathcal{H} \equiv [j_1] \otimes [j_2]$ is considered. This leads to the reparametrisation of the spin-density operator. The new parameters (polarisation tensors) are related to the mean values of the operators built up from the individual spin operators (5) which could sometimes be more convenient for theoretical considerations (in connection with e.g. the depolarisation processes) or from the experimental point of view. If the spin-density operator is diagonal in the total spin then the new parameters are no longer independent. This is the source of the linear relations between components of the polarisation tensors and the simple formula which describes all such identities is derived.

Throughout the paper the examples, most important from the point of view of possible applications to the phenomenological analysis of some nuclear reactions, including the nuclear muon-capture reaction by oriented or polarised targets with *arbitrary* non-zero spins, are considered in detail. Towards the end of the paper we discuss the simplest model of the depolarisation due to the spin–spin hyperfine interactions.

2. Tensor operators

We start with the basis-independent definition of the tensor operator. Let $\mathcal{L}$ and $\mathcal{H}$ be the pair of linear spaces. Quite generally the tensor operator of the *rank* $\mathcal{L}$ is the linear mapping

$$\mathcal{L} \to \mathrm{End}\,\mathcal{H} \simeq \mathcal{H} \otimes \mathcal{H}^*. \tag{6}$$

This means that the space of rank $\mathcal{L}$ tensor operators in $\mathcal{H}$ is $(\mathrm{End}\,\mathcal{H}) \otimes \mathcal{L}^*$, where $\mathcal{L}^*$ denotes the dual space of $\mathcal{L}$. Usually, however, the tensor operators are defined with respect to some group G. This is the particular case of (6) if both spaces, $\mathcal{L}$ and $\mathcal{H}$, are the carrier spaces of the two representations of G. Then the tensor operators with respect to G are defined as the G-*invariant* mappings (6). Suppose that $\mathcal{H}$ decomposes into the direct sum of the irreducible, with respect to G, subspaces $\mathcal{H} = \oplus_F \mathcal{H}_F$ (for simplicity we consider the case without the multiplicities, $m_F = 1$). This splitting can conveniently be represented uniquely through the set of the projectors $\{\mathrm{P}_F\}$, i.e. the idempotent operators, such that

$$\sum_F \mathrm{P}_F = \mathrm{i}d_{\mathcal{H}} \qquad \text{and} \qquad \mathrm{P}_F \circ \mathrm{P}_E = \delta_{FE}\,\mathrm{P}_F. \tag{7}$$

Now the representation of the group G in $\mathcal{H}$ preserves the set of the operators $\{\mathrm{P}_F\}$. In other words, the P_F operator for each value of the index F is an invariant (scalar) operator with respect to the G action in $\mathcal{H}$. The particular case of the above situation occurs when the index F takes only one value, i.e. when $\mathcal{H}$ itself is the carrier space of the irreducible representation of G.

Next we introduce the rank $\mathcal{L}$ Wigner tensor operators $\{\mathcal{P}_{FE}\}$ with respect to the given G-irreducible splitting $\{P_F\}$ of $\mathcal{H}$. Then the arbitrary (not Wigner) tensor operator of the same rank and corresponding to the same pair $\mathcal{H}_F$ and $\mathcal{H}_E$ of the subspaces of $\mathcal{H}$ is simply the arbitrary factor times the Wigner tensor operator. This factor, which depends on $\mathcal{L}$, $\mathcal{H}_F$ and $\mathcal{H}_E$ spaces (and does *not* depend obviously on any basis in these spaces), usually is referred to as the 'reduced matrix element' (cf Klimyk 1983). The $\mathcal{P}_{FE}$ Wigner tensor operator is *defined* as follows

$$\mathcal{P}_{FE} = P_F \circ P_E \otimes id_{\mathcal{L}} \qquad \mathcal{L} \to \operatorname{End} \mathcal{H}. \tag{8}$$

The G-invariance of the $\{\mathcal{P}_{FE}\}$ tensor operators is self evident. If $v \in \mathcal{L}$ then

$$\mathcal{P}_{FE}(v) = P_F \circ P_E \otimes v \in \operatorname{End} \mathcal{H} \tag{9}$$

is referred to as the v-component of $\mathcal{P}_{FE}$. Definition (8) is equivalent to the Wigner-Eckart relation. In order to see this let $\{|F^\mu\rangle\}$ be the arbitrary basis in $\mathcal{H}_F$ and $\{\langle F^\mu|\}$ be the corresponding dual basis in $\mathcal{H}_F^*$. Then

$$P_F = \sum |F^\mu\rangle \otimes \langle F^\mu|. \tag{10}$$

Therefore

$$\langle F^\mu|\mathcal{P}_{FE}(v)|E^\nu\rangle = \langle F^\mu|(|E^\nu\rangle \otimes v), \tag{11}$$

i.e. the matrix elements of any component of the Wigner tensor operators coincide with the Clebsch-Gordan coefficients. Evidently the tensor operator does *not* depend on the basis either in $\mathcal{L}$ or in $\mathcal{H}$ spaces (cf Klimyk 1983). The Wigner-Eckart relation (11) shows that the $\mathcal{P}_{FE}$ tensor operator (8) coincides up to the phase with that of $W_0(E, F)$ operator of Klimyk (1983, definition (21)).

The tensor operator of rank $\mathcal{L}$ is called *irreducible* if the space $\mathcal{L}$ is the carrier space of the irreducible representation of G.

Suppose that the $\mathcal{L}$ space decomposes into the direct sum of the G-irreducible subspaces $\mathcal{L} = \oplus_L \mathcal{L}_L$. Let P^L denote the corresponding projector operator, $P^L: \mathcal{L} \to \mathcal{L}_L$, then the operator

$$\mathcal{P}^L_{FE} \equiv P_F \circ P_E \otimes P^L \tag{12}$$

is the G-irreducible tensor operator. It should be obvious that the set of the tensor operators (12) spans the particular basis in End $\mathcal{H}$. We will discuss a further important basis in End $\mathcal{H}$ in § 7.

The tensor products of the tensor operators are the mappings $\mathcal{L} \otimes \mathcal{L} \to \operatorname{End} \mathcal{H}$, etc. Corresponding to the irreducible decomposition $\mathcal{L}_L \otimes \mathcal{L}_K = \oplus \mathcal{L}^R_{LK}$, one can derive the Clebsch-Gordan decompositions of the tensor products of the tensor operators.

For the case of the rotation group SO(3), where $\mathcal{L}_L \simeq [L]$ denotes the (2L+1)-dimensional irreducible representation (and the other indices have a similar meaning) the decomposition of the tensor product of the Wigner tensor operators (12) has the form

$$\mathcal{P}^L_{FA} \underset{\mathcal{L}}{\otimes} \mathcal{P}^K_{BE} = \sum (-)^{K+A-F}(2A+1)(2R+1) \begin{Bmatrix} L & K & R \\ F & E & R \\ A & B & O \end{Bmatrix} \mathcal{P}^R_{FE} \circ P^R_{LK}. \tag{13}$$

Here { } denotes the $9j$ symbol, known as the Fano coefficient (see Fano 1951, Fano and Racah 1959, Varschalovich *et al* 1975 or Jucys and Bandazitis 1977) and P^R_{LK} is the projector on $\mathscr{L}^R_{LK}$.

The G group representations in the space $\mathscr{L}$ determine the natural G-invariant isomorphism $\phi: \mathscr{L} \to \mathscr{L}^*$ which is defined by means of the Clebsch-Gordan coupling to the one-dimensional (scalar) representation of G. G-invariance means that $g^* \circ \phi = \phi \circ g^{-1}$, where g^* denotes the pull-back of g. For the SO(3) case we choose the following convention

$$(\phi v)w = (-)^L \hat{L} \langle 0|v \otimes w\rangle \qquad \text{for } v \text{ or } w \in [L]. \tag{14}$$

Here $\hat{L} \equiv (2L+1)^{1/2}$ and $\langle 0|v \otimes w\rangle$ denotes the usual Clebsch-Gordan coefficient of the SO(3) group. The above isomorphism ϕ induces the G-invariant bilinear mappings (the scalar products) symmetric for integer values of L:

$$\begin{aligned} &\underset{\sim}{\phi}: \mathscr{L} \times \mathscr{L} \ni v, w \to \underset{\sim}{\phi}(v, w) \equiv (\phi v) w, \\ &\tilde{\phi}: \mathscr{L}^* \times \mathscr{L}^* \ni \alpha, \beta \to \tilde{\phi}(\alpha, \beta) \equiv \alpha(\phi^{-1}\beta). \end{aligned} \tag{15}$$

As $\tilde{\phi} \in \mathscr{L} \otimes \mathscr{L}$, then from (13) one can calculate

$$(\mathscr{P}^L_{FE} \otimes \mathscr{P}^K_{E'F'})\tilde{\phi} = \delta^{LK} \delta_{FF'} \delta_{EE'} (-)^{F-E} (\hat{E}/\hat{F}) \mathrm{P}_F. \tag{16}$$

The left-hand side of the above formula will be referred to as the dot product of the irreducible tensor operators (the scalar product with respect to the $\mathscr{L}$ space, induced by ϕ (14)) and will be denoted by

$$\mathscr{P}^L_{FE} \cdot \mathscr{P}^K_{E'F'} \equiv (\mathscr{P}^L_{FE} \otimes \mathscr{P}^K_{E'F'})\tilde{\phi}. \tag{17}$$

It should be obvious that the above G-invariant dot product is not generally symmetric, the tensor operators do not commute in the $\mathscr{H}$ space. Generally the G-invariant dot product

$$M \cdot N \equiv (M \otimes N)\tilde{\phi} \tag{18}$$

is well defined for M and $N: \mathscr{L} \to \mathrm{End}\, \mathscr{H}$ *or* $\mathbb{C}$, cf (15).

For the multipole expansion of the spin-density operator we will need the trace formula for SO(3):

$$\mathrm{Tr}\{\mathscr{P}^L_{FE} \otimes \mathscr{P}^K_{E'F'}\} = (-)^{F-E} \frac{\hat{F}\hat{E}}{\hat{L}\hat{K}} \delta_{FF'} \delta_{EE'} \underset{\sim}{\phi} \circ \mathrm{P}^L \otimes \mathrm{P}^K \tag{19}$$

and the completeness relation

$$\sum_{FEL} (-)^{E-F} \frac{\hat{L}\hat{L}}{\hat{F}\hat{E}} \langle A|\mathscr{P}^L_{FE}|B\rangle \cdot \langle C|\mathscr{P}^L_{EF}|D\rangle = \langle A|D\rangle\langle C|B\rangle. \tag{20}$$

In (19)-(20) $\hat{F} \equiv (2F+1)^{1/2}$, etc; also the dot product in (20) is understood in the sense of the definitions (15) and (18): $\alpha \cdot \beta \equiv \tilde{\phi}(\alpha, \beta)$.

3. Polarisation operators and the Lie algebra

The polarisation operators are a special kind of tensor operator and can be defined quite generally for arbitrary Lie group G as the subset of the tensor operators, i.e. the G-invariant mappings (6), which commute with the set of all Casimir operators of G.

The most important example of the polarisation operators is the representation of the Lie algebra of G itself in the space $\mathscr{H}$. The Casimir operators in $\mathscr{H}$ can also be considered as the polarisation operators whose *values* on $\mathscr{L}$ are trivially G-invariant operators. We shall identify the Lie algebra $\mathscr{G}$ of the Lie group G with the space of the adjoint representation of G. Then the polarisation operators corresponding to the Lie algebra of G are the G-invariant mappings $\mathscr{G}\rightarrow \mathrm{End}\,\mathscr{H}$. The rank of these Lie algebra polarisation operators is determined by the G-irreducible decomposition of the $\mathscr{G}$ (the case of the non-decomposable adjoint representation of G needs a separate analysis which we omit here). Then one can build up the Lie algebra polarisation operators of *arbitrary* rank using the tensor products of the Lie algebra polarisation operators of 'the lowest rank' and the Clebsch–Gordan decomposition. Corresponding to the G-irreducible decompositions

$$l\equiv \overset{L}{\otimes}\,\mathscr{G}=\oplus\,\mathscr{L}_K \qquad \text{and} \qquad \mathscr{H}=\oplus\,\mathscr{H}_F$$

the rank L Lie algebra polarisation operator has the general form

$$\mathscr{F}^L=\sum_F c_F^L\mathscr{P}_F^L:\qquad \mathscr{L}_L\rightarrow \mathrm{End}\,\mathscr{H}, \tag{21}$$

where $\mathscr{P}_F^L\equiv\mathscr{P}_{FF}^L$ as given by (12). The numbers c_F^L ('reduced matrix elements') are exactly 'the new invariants of the irreducible representation of G in $\mathscr{H}_F$' discovered recently by Klimyk (1983). As should be evident, and this was also pointed out by Klimyk, the number of such independent invariants, for the fixed $\mathscr{H}_F$ space, is equal to the number of the independent Casimir operators. We do not agree with Klimyk's statements that the *meaning* of these invariants in the representation theory is not clear. They are rather simply related to the eigenvalues of the Casimir operators. The compactness of G seems to be irrelevant.

The Lie algebra-polarisation operator for the rotation group SO(3) is the rank 1 ($L=1$ in (2.1)) tensor operator $\mathscr{F}$, to which we will refer to as the total spin operator in $\mathscr{H}$. The only Casimir operator has the form (using (16)–(17))

$$\mathscr{F}\cdot\mathscr{F}=\sum_F (c_F^1)^2P_F\Rightarrow(c_F^1)^2=F(F+1). \tag{22}$$

Both formulae (21) and (22) lead to the well known expression for the matrix elements of the generators of SO(3) (Rose 1957, 1962).

The $\mathscr{P}_F^L$ operators span the basis of the irreducible polarisation operators in $\mathscr{H}$ because

$$[\mathscr{P}_{FE},\mathscr{F}\cdot\mathscr{F}]=0\Leftrightarrow F=E.$$

The aim of the rest of this section is to build up explicitly for the SO(3) group the polarisation operators of *arbitrary* rank L in terms of the total spin-$\mathscr{F}$, rank-1 operator. We *define* the rank-L, spin-F ppolarisation operator $\mathscr{F}_F^L\equiv\mathscr{F}^L\circ \mathrm{P}_F=\mathrm{P}_F\circ\mathscr{F}^L$ through the recurrent relation

$$\mathscr{F}_F^L=(\mathscr{F}_F^{L-1}\otimes\mathscr{F}_F)^L\equiv(f_F^{L-1}\otimes f_F). \tag{23}$$

Now

$$\mathscr{F}^L=\sum_F c_F^L\mathscr{P}_F^L:\overset{L}{\otimes}\,\mathscr{G}\rightarrow \mathrm{End}\,\mathscr{H} \tag{24}$$

where $\mathscr{G}$ is the Lie algebra of SO(3), cf (21). Using (13) one can get† for $L \geqslant 1$

$$4(2L-1)(c_F^L)^2 = L(2F+1-L)(2F+1+L)(c_F^{L-1})^2 \qquad \text{with } c_F^0 = 1. \tag{25}$$

We will refer to the $\mathscr{F}^L$ operator, given by (24)-(25), as the rank-L total spin operator in $\mathscr{H}$.

We would like to stress the clear conclusion which follows from these considerations. We have described the *two* particular bases of the irreducible polarisation operators in $\mathscr{H}$: the first one consists of the $\{\mathscr{P}_F^L\}$ Wigner polarisation operators. Instead of them one can use any other basis, cf the basis adopted by Werle (1966) and also that adopted by Csonka *et al* (1966) and Steiger and Fritz (1967). What is essential is the relation of the operators which span the *basis* of the irreducible polarisation operators in $\mathscr{H}$ to the representation of the Lie algebra which has direct physical meaning. In the case considered of the SO(3) group, we have built up the basis $\{\mathscr{F}_F^L\}$ (23) of the irreducible polarisation operators in $\mathscr{H}$, in terms of the total spin operator $\mathscr{F}$ and the set of projectors $\{P_F\}$. Another physically relevant basis will be considered in the § 7, in terms of the spin operators of the subsystems.

It should be obvious that for the case when the space $\mathscr{H}$ is the carrier space of the *irreducible* representation of G, the polarisation operators coincide with the G-irreducible tensor operators. This case has been discussed exhaustively e.g. by Varschalovich *et al* (1975).

4. Multipole expansion of the spin-density operator in terms of the total spin operator

From the trace formula (19) and the completeness relation (20) one can easily show that the multipole expansion of any operator $\rho \in \operatorname{End} \mathscr{H}$ has the form

$$\rho = \sum_{FEL} (-)^{F-E} \frac{2L+1}{\hat{F}\hat{E}} \langle \mathscr{P}_{EF}^L \rangle \cdot \mathscr{P}_{FE}^L, \tag{26}$$

where

$$\langle \mathscr{P}_{EF}^L \rangle \equiv \operatorname{Tr}\{\rho \mathscr{P}_{EF}^L\} \in \mathscr{L}_L^*. \tag{27}$$

Let us consider the spin-density operator diagonal in the total spin,

$$[\rho, \mathscr{F}\cdot\mathscr{F}] = 0 \Leftrightarrow [\rho, P_F] = 0. \tag{28}$$

This is equivalent to saying that in (26)

$$\langle \mathscr{P}_{EF}^L \rangle = \delta_{FE} \langle \mathscr{P}_F^L \rangle. \tag{29}$$

This means that the spin-density operator is itself the polarisation operator in $\mathscr{H}$, i.e. ρ describes the incoherent mixture of the states with the definite total channel spin F. Usually we refer to the mean values of the polarisation operators i.e. to $\langle \mathscr{P}_F^L \rangle$ (27), as the Fano polarisation multipole parameters or as the statistical tensors (see Fano 1951).

For the properly normalised vector $v \in \mathscr{L}_L$, the trace formula (19) gives (one should put $K = 0$ in (19))

$$\operatorname{Tr}\{\mathscr{P}_{FE}^L(v)\} = \delta_{FE}(2F+1)\delta^{LO}.$$

† There is a misprint in the corresponding formula (38.18) in Jucys and Bandzaitis (1977).

Moreover $\mathscr{P}^O_{FE}(w)=\delta_{FE}\mathrm{P}_F$ for $\phi(w,w)=1$ and $w\in\mathscr{L}_O$, therefore

$$\mathrm{Tr}\,\rho=\sum_F p_F=1, \qquad \text{where } p_F\equiv\langle\mathrm{P}_F\rangle \tag{30}$$

is the *population* of the F-states. Using (28)-(29) and (19) we have

$$\mathrm{Tr}\,\rho^2=\sum_{FL}\frac{2L+1}{2F+1}\langle\mathscr{P}^L_F\rangle\cdot\langle\mathscr{P}^L_F\rangle. \tag{31}$$

This can be presented as

$$\left(\sum_F\frac{p_F^2}{2F+1}\right)+\sum_{F,L\geq1}\frac{2L+1}{2F+1}\langle\mathscr{P}^L_F\rangle\cdot\langle\mathscr{P}^L_F\rangle\leq1.$$

We introduce the function g of the populations $(0\leq g<1)$

$$g\equiv\left\{1-\sum_F\frac{p_F^2}{2F+1}\right\}^{1/2}, \qquad g=0\Leftrightarrow p_F=\delta_{F,0}. \tag{32}$$

Then one can *define* the non-negative degrees of the orientation λ^L_F, $0\leq\lambda^L_F\leq1$, of rank $L\geq1$ as follows

$$\hat{L}\langle\mathscr{P}^F_F\rangle\equiv g\hat{F}\lambda^L_F e^L_F, \tag{33}$$

where $e^L_F\in\mathscr{L}^*_L$ are the unit covectors: $e^L_F\cdot e^L_F=1$, and

$$\sum_{F,L\geq1}(\lambda^L_F)^2\leq1. \tag{34}$$

In the following it is convenient to introduce the ρ-dependant operators

$$\mathrm{P}^L_F\equiv e^L_F\cdot\mathscr{P}^L_F=\mathscr{P}^L_F(\boldsymbol{e}^L_F) \qquad \text{and} \qquad F^L_F\equiv e^L_F\cdot\mathscr{F}^L_F=\mathscr{F}^L_F(e^L_F)=\boldsymbol{e}^L_F\cdot\boldsymbol{F}^L_F \tag{35}$$

where $\phi\boldsymbol{e}^L_F=\boldsymbol{e}^L_P$ and $\mathscr{F}^L_F=\boldsymbol{F}^L_F\circ\phi$. Then

$$\hat{L}\langle F^L_F\rangle=g\hat{F}c^L_F\lambda^L_F. \tag{36}$$

In (35) $\mathrm{P}^0_F=\mathrm{P}_F$ and in (36) the λ^0_F parameters are defined as follows

$$g\hat{F}\lambda^0_F=p_F \qquad \text{for } p_F\neq\delta_{F,0}. \tag{37}$$

The multipole expansion of the spin-density operator in terms of the total spin operators $\mathscr{F}^L$ (23)-(25) and of the projectors $\{\mathrm{P}_F\}$ can be obtained simply by substituting $\mathscr{P}^L_F\to\mathscr{F}^L_F=c^L_F\mathscr{P}^L_F$ into (26) with (29). However, for the calculations of the angular and the polarisation distributions in nuclear and elementary particle reactions, see for instance Werle (1966); it is more convenient to have the multipole expansion in terms of the Wigner polarisation operators $\mathscr{P}^L_F$ (8), i.e. in terms of P^L_F (35), which matrix elements coincide with the Clebsch-Gordan coefficients. On the other hand for the fit to the experimental data it is necessary to have the relationship between all the parameters entering the spin-density operator and the measurable physical observables, i.e. to the mean values of the physically meaningful operators which are $\mathscr{F}^L_F$ operators rather than $\mathscr{P}^L_F$. Therefore for the phenomenological analysis and for the device of the experimental measurements the most essential is (36) which determines the degrees of the orientation in terms of the mean values of the total spin operators $\mathscr{F}^L$ (23)-(25).

To summarise, we arrive at the final, simple form of the multipole expansion of the spin-density operator (we assume hereafter that $p_F\neq\delta_{F,0}$)

$$\rho=g\sum(\hat{L}/\hat{F})\lambda^L_F\mathrm{P}^L_F. \tag{38}$$

It is important that the λ_F^L parameters in (36) and (38) describing the populations and the degrees of the orientation are the non-negative real numbers obeying the conditions (30), (34) and (37).

One of the most essential conclusions which follows from the above multipole expansion (38) is that the spin-density operator ρ can *not* be presented in the factorised form (4) as we discussed in § 1. This is due to the population-dependent g function (32) which enters into (33), (36) and (38). In (36)–(38) the populations $\{p_F\}$ and the degrees of the orientation $\{\lambda_F^L, L \neq 0\}$ are *truly independent* parameters as follows from (34).

Let us investigate briefly the function g defined by means of (32). The conditional critical (extremal) points of g on the surface (30) are the solutions of the system

$$\left.\begin{aligned} dg \wedge d \sum_F p_F = 0 \\ \sum_F p_F = 1 \end{aligned}\right. \Rightarrow p_F = \overset{\text{stat}}{p_F} \equiv \frac{\dim \mathscr{H}_F}{\dim \mathscr{H}} = \frac{2F+1}{\Sigma_E\,(2E+1)}. \tag{39}$$

This shows that in the space of the populations $\{p_F\}$ the only critical point of g is the maximum and it is given by the statistical weights $\{p_F^{\text{stat}}\}$. Therefore

$$g \leqslant \overset{\text{stat}}{g} \equiv \{1 - 1/\dim \mathscr{H}\}^{1/2}. \tag{40}$$

Until now we have not specified the space $\mathscr{H}$ of the linear representation of the SO(3) group. Some of the previous results obviously could also be generalised for the arbitrary Lie group. If the space $\mathscr{H}$ is the carrier space of the *unitary* representation of SO(3) then the invariant spin-F operators F_F^L, as well as P_F^L, (35), are *Hermitian* operators which ensures among other things that $\rho = \rho^+$. In this respect compare with the discussion by Werle (1966). Many more limitations for the possible direct applications of the multipole expansion (38) come from the SO(3)-irreducible decomposition $\mathscr{H} = \oplus_F \mathscr{H}_F$. In fact we do not take into account the possible multiplicities of this decomposition; for instance the multiplicities will essentially modify the basis of the polarisation operators and all subsequent formulae in this section. Therefore, the multipole expansion (38) can be applied directly only to the case when the decomposition $\mathscr{H} = \oplus_F \mathscr{H}_F$ does not contain the multiple irreducible repressentations of SO(3). This is the case when $\mathscr{H}$ is itself the space of the irreducible representation $\mathscr{H} = [j]$, or when $\mathscr{H} = [j_1] \otimes [j_2]$.

In this way the spin-density operator diagonal in the total spin is completely parametrised by means of the populations $\{p_F\}$, (30), the degrees of the orientation $\{\lambda_F^L, L \neq 0\}$, (36) with (34), and of the unit polarisation tensors (directions) $\{e_F^L\}$, (33) and (35). This parametrisation has been summarised in the introduction for the case $\mathscr{H} = [j_1] \otimes [j_2]$, see (1)–(3).

The case $\mathscr{H} = [j]$ has been rather exhaustively presented by Werle (1966) and in the other papers cited in § 1. In this case one should put $p_F = \delta_{Fj}$, $j \neq 0$, in (32) which gives

$$g = \{2j/(2j+1)\}^{1/2}, \tag{41}$$

and the multipole expansion (38) of the spin-density operator $\rho(j)$ takes the form

$$\rho(j) = \frac{1}{2j+1}\left\{\mathrm{i}d + (2j)^{1/2} \sum_{L \geqslant 1} \hat{L} \lambda_j^L P_j^L\right\}, \tag{42}$$

where $\hat{L}\langle F_j^L\rangle = (2j)^{1/2} c_j^L \lambda_j^L$. Needless to say that, for example, $\rho(1)$ describes the completely polarised pure state iff $(\lambda_1^1)^2 + (\lambda_1^2)^2 = 1$, i.e. that for the pure polarisation state the degree of the vector, rank 1, orientation λ_j^1 need not to be 1 for spin $j \geq 1$. For the case of the axial symmetry see (80)-(82).

In connection with the multipole expansion (42) we would like to add the following remark. In spite of the fact that the vector polarisation, $L=1$ term in (42) is correctly calculated by Werle (1966), some authors still refer to and use the incorrect expression presented in the well known textbook by Schiff (1968). To see this let us write the first-order multipole in (42) explicitly (cf also with (47) and (77))

$$\rho(j) = \frac{1}{2j+1}\left\{ id + \left(\frac{6}{j+1}\right)^{1/2} \lambda_j^1 e_j^1 \cdot \mathscr{F}_j + (2j)^{1/2} \sum_{L \geq 2} \hat{L} \lambda_j^L \mathrm{P}_j^L \right\}. \tag{42a}$$

The above formula, wherein $0 \leq \lambda_j^L \leq 1$ and $e_j^1 \cdot e_j^1 = 1$, has been derived by Werle (1966). Comparing this with the Schiff expression (Schiff 1968, p 381)

$$\rho(j) = \frac{1}{2j+1}\left\{ id + \frac{3}{j+1} \mathscr{E} \cdot \mathscr{F}_j + (\text{higher order multipoles}) \right\},$$

we obtain $|\mathscr{E}| \leq [\frac{2}{3}(j+1)]^{1/2}$, contrary to the incorrect claim that $|\mathscr{E}| \leq 1$. The Schiff formula has been adopted by Hambro and Mukhopadhyay (1975) and also by Mukhopadhyay (1977). Therefore most of the qualitative and quantitative conclusions for the target nuclear spin $j \neq \frac{1}{2}$ presented by Hambro and Mukhopadhyay (1975) are incorrect. In fact the maximal values of $|\mathscr{E}|$, in the Hambro and Mukhopadhyay notation, are $\frac{2}{3}3^{1/2}$, $(\frac{5}{3})^{1/2}$ and $2(\frac{2}{3})^{1/2}$ for the target nuclear spin $j = 1, \frac{3}{2}$, and 3 respectively.

The case $\mathscr{H} = [j_1] \otimes [j_2]$, in which we are mostly interested in this paper, will be discussed in some details in § 5. To consider spin-spin and spin-orbit interactions (Djrbashyan 1959, Shmushkevich 1959) for the non-relativistic system of the two spins j_1 and j_2 on the definite shell one should take $\mathscr{H} = [j_1] \otimes [j_2] \otimes [l]$. For this case which is important in the theory of the atomic depolarisation, one should rederive the multipole expansion of the spin-density operator taking into account the multiplicities $\{m_F\}$ of the irreducible decomposition $\mathscr{H} = \oplus_F m_F \mathscr{H}_F$, along the line presented e.g. by Klimyk (1983). However, this is outside of the scope of the present paper.

5. Examples for $\rho(j_1 \otimes j_2)$

We will consider in some detail the case $\mathscr{H} = [j_1] \otimes [j_2]$ of the interacting system of the two spins j_1 and j_2 on the K-shell, when $[\rho \mathrm{P}_F] = 0\ \forall F$. Let us consider the population dependent g function (32). Because $\dim \mathscr{H} = (2j_1+1)(2j_2+1)$ then one can show, using (40), that

$$\left\{\frac{2|j_1 - j_2|}{2|j_1 - j_2| + 1}\right\}^{1/2} \leq g \leq \left\{\frac{4j_1 j_2 + 2(j_1 + j_2)}{(2j_1+1)(2j_2+1)}\right\}^{1/2} \equiv \overset{\text{stat}}{g}. \tag{43}$$

The most important is the case when $\mathscr{H} = [\frac{1}{2}] \otimes [j]$. In this case we have the two hyperfine states $F_\pm \equiv j \pm \frac{1}{2}$. These states will be characterised by the quantities with the subscript '+' and '−' respectively. As $p_+ + p_- = 1$, then the g-function (32) can be presented in two alternative forms (for $j \neq 0$):

$$g^2 = \frac{4j+1}{4j+2} - \frac{1}{4j+2} \cdot \frac{j+1}{j}\left(1 - \frac{2j+1}{j+1} p_+\right)^2$$
$$= \frac{4j+1}{4j+2} - \frac{1}{4j+2} \cdot \frac{j}{j+1}\left(1 - \frac{2j+1}{j} p_-\right)^2. \qquad (44)$$

From (43) for $j \neq 0$ we have

$$\left\{\frac{2j-1}{2j}\right\}^{1/2} \leqslant g \leqslant \left\{\frac{4j+1}{4j+2}\right\}^{1/2}, \qquad (45)$$

and for $j = 0 \Rightarrow g = 2^{-1/2}$. By denoting $p \equiv p_+$, from (44) we have in particular

$$\begin{aligned} &j = \tfrac{1}{2} \Rightarrow g = \{2p - \tfrac{4}{3}p^2\}^{1/2}, \qquad p \neq 0 \text{ (cf with (37))}, \\ &j = 1 \Rightarrow g = \{\tfrac{1}{2} + p - \tfrac{3}{4}p^2\}^{1/2} \\ &j = \tfrac{3}{2} \Rightarrow g = \{\tfrac{2}{3}(1 + p - \tfrac{4}{5}p^2)\}^{1/2}. \end{aligned} \qquad (46)$$

Let us consider the systems with vector polarisations only, i.e. when $\langle F_F^L \rangle = 0$ for $L \geqslant 2$. This happens, for example, in the case of the capture of polarised spin-$\frac{1}{2}$ muons by unpolarised spin-j nuclei. The spin-density operator $\rho(\frac{1}{2} \otimes j)$ for this process has been considered by Bukhvostov and Popov (1967, 1970). In order to compare this with the Bukhvostov and Popov results one should substitute $P_F^L \to F_F^L = c_F^L P_F^L$ (35) into the multipole expansion (38). Now putting $\lambda_F \equiv \lambda_F^1$, we get from (38) using (37) and (25)

$$\rho = \sum_F \left(\frac{p_F}{2F+1} P_F + g\frac{3^{1/2}}{\hat{F}} \frac{\lambda_F}{[F(F+1)]^{1/2}} F_F\right). \qquad (47)$$

The degrees of the vector polarisations λ_F are defined through the mean values of the total spin operators according to (36). In the full line this reads

$$\hat{F} g \lambda_F \equiv [3/F(F+1)]^{1/2} \langle F_F \rangle. \qquad (48)$$

The spin-density operator (47) is parametrised by the populations $\{p_F\}$, (30), and the degrees of the vector polarisations $\{\lambda_F\}$, such that

$$0 \leqslant p_F, \qquad \lambda_F \leqslant 1 \qquad \text{and} \qquad \sum_F p_F = 1, \qquad \sum_F (\lambda_F)^2 \leqslant 1. \qquad (49)$$

They do *not* describe the spin-density operator (47) completely. The rest of the independent information is contained in the set of the directions of the polarisation $\{\boldsymbol{e}_F \equiv \boldsymbol{e}_F^1\}$ of the F-states,

$$F_F = P_F \circ \mathscr{F}, \qquad \boldsymbol{e}_F \cdot \boldsymbol{e}_F = 1, \qquad (50)$$

cf (35) and (21).

Let us now adapt (47) and (48) to the case $\mathscr{H} = [\frac{1}{2}] \otimes [j]$,

$$\rho(\tfrac{1}{2} \otimes j) = \frac{1}{2(j+1)} p_+ P_+ + \frac{1}{2j} p_- P_-$$
$$+ g\left(\frac{6}{2j+1}\right)^{1/2} \left(\frac{\lambda_+ F_+}{[(j+1)(2j+3)]^{1/2}} + \frac{\lambda_- F_-}{[j(2j-1)]^{1/2}}\right), \qquad (51)$$

where

$$g\lambda_+ \equiv \left(\frac{6}{(j+1)(2j+1)(2j+3)}\right)^{1/2} \langle F_+ \rangle, \qquad g\lambda_- \equiv \left(\frac{6}{j(2j-1)(2j+1)}\right)^{1/2} \langle F_- \rangle. \tag{52}$$

The comparison of the general multipole expansion (51) with formula (9) in Bukhvostov and Popov (1967) (or with expansion (7) in Bukhvostov and Popov (1970)) gives the relation of our non-negative degrees of the vector polarisation λ_F, $\lambda_+^2 + \lambda_-^2 \leqslant 1$, to the Bukhvostov-Popov (BP) parameters λ_F^{BP},

$$p_F \lambda_F^{\mathrm{BP}} = \frac{4(F-j)}{2j+1} \{\tfrac{1}{3}(2F+1)F(F+1)\}^{1/2} g\lambda_F \varepsilon_F,$$

or in the full line

$$p_+ \lambda_+^{\mathrm{BP}} = g\left(\frac{2}{3}\,\frac{(j+1)(2j+3)}{2j+1}\right)^{1/2} \lambda_+ \varepsilon_+,$$

$$p_- \lambda_-^{\mathrm{BP}} = g\left(\frac{2}{3}\,\frac{j(2j-1)}{2j+1}\right)^{1/2} \lambda_- \varepsilon_-, \tag{53}$$

where $\varepsilon_F \equiv \boldsymbol{a} \cdot \boldsymbol{e}_F$ (cf (67) below) and $\boldsymbol{a}$ denotes the unit vector, $\boldsymbol{a} \cdot \boldsymbol{a} = 1$, of the axial symmetry of the spin-density operator. It should be stressed that the Bukhvostov and Popov parameters $\{\lambda_F^{\mathrm{BP}}\}$ are defined (see (119)) only for the case of axial symmetry which we will consider in detail in § 6. We should be aware, however, that generally in (51)-(52) $\boldsymbol{e}_+ \cdot \boldsymbol{e}_- \neq 1$. The g-function in (51)-(53) is given by (44). In (53) the axial symmetry implies $\varepsilon_F = \pm 1$.

The maximal values of $|p_F \lambda_F^{\mathrm{BP}}|$ correspond to the statistical weights, (45), see also (39) and (43). The bounds for λ_F^{BP} derived by Bukhvostov and Popov (1967) follow from the particular model of the depolarisation which we consider in § 8 (compare (53) with the model-dependent formulae (143)-(144)). One should note also that formulae (9) and (11) in Bukhvostov and Popov (1967) are not compatible because the weight factors p_F have been missed out in (11). The misprint has been corrected in Bukhvostov and Popov (1970), however it appeared again in Bukhvostov *et al* (1972, formulae (4) and (7)). These misprints do *not* disturb of any of the results presented there, however they are relevant with respect to the fit to the experimental data (cf for instance Bukhvostov *et al* 1971)). We discuss the definition of the Bukhvostov and Popov parameters $\{\lambda_F^{\mathrm{BP}}\}$ at the end of § 7 see (118).

The explicit expressions for the few first multipoles of (38). The c_F^L-coefficients (25) which enter into (36) must be calculated explicitly; they define the rank-L total spin operator $\mathscr{F}^L$, (21), (23)-(24). Also from (35) we have

$$F_F^L = c_F^L \mathrm{P}_F^L. \tag{54}$$

Quite generally (36) takes the form

$$2^{L/2} \hat{L} \langle F_F^L \rangle = L! \left\{ \frac{(2F+1+L)!}{(2L)!(2F-L)!} \right\}^{1/2} g\lambda_F^L. \tag{55}$$

In particular for the few first multipoles

$$\begin{aligned}
&\langle F^0_F\rangle = \hat{F}g\lambda^0_F\\
&3^{1/2}\langle F^1_F\rangle = \hat{F}\{F(F+1)\}^{1/2}g\lambda^1_F\\
&30^{1/2}\langle F^2_F\rangle = \hat{F}\{F(F+1)(2F-1)(2F+3)\}^{1/2}g\lambda^2_F\\
&(70)^{1/2}\langle F^3_F\rangle = \hat{F}\{F(F+1)(2F-1)(2F+3)(F-1)(F+2)\}^{1/2}g\lambda^2_F\\
&3(70)^{1/2}\langle F^4_F\rangle = \hat{F}\{F(F+1)(2F-1)(2F+3)(F-1)(F+2)(2F-3)(2F+5)\}^{1/2}g\lambda^4_F.
\end{aligned} \tag{56}$$

The spin-density operator $\rho(\frac{1}{2}\otimes j)$ *without* the restrictive assumption of the vector polarisation, i.e. when $\langle F^{L>1}\rangle \neq 0$, has been considered firstly by Bukhvostov and Popov (1964) for $j=\frac{1}{2}$ and by Bukhvostov *et al* (1972) for $j=1$. Later this spin-density operator was considered for the arbitrary spin-j of the nuclear target by Hambro and Mukhopadhyay (1975) and discussed by Mukhopadhyay (1977), for the particular simple model of the depolarisation due to the spin-spin hyperfine interaction on the K-shell only. This model is considered in §§ 8 and 9 where we generalise some of the results due to Hambro and Mukhopadhyay. In all these papers the factorised form (4) has been adopted. Equation (55) allows us to write down explicitly the multipole expansion (38) of the spin-density operator $\rho(j_1\otimes j_2)$ for completely *arbitrary* spins j_1 and j_2.

6. Cylindrical symmetry

Let $s\in\mathcal{G}^*$, such that $\tilde{\phi}(s,s)\equiv s\cdot s=1$ (see (15)), describes the *direction* of the axial symmetry of the spin-density operator ρ. Here $\mathcal{G}$ is the Lie algebra of the SO(3) group (see § 3). For the total spin operator $\mathcal{F}$, (21)-(22), in the space $\mathcal{H}$, we define the *operator*

$$m\equiv\mathcal{F}(s)\in \operatorname{End}\mathcal{H}. \tag{57}$$

Then the cylindrical symmetry of the spin-density operator ρ means that

$$[\rho, m]=0. \tag{58}$$

Inserting the multipole expansion (38) into the above equation and using (11)-(12) we can easily obtain the equivalent equation for ε^L_F:

$$\mathcal{P}^L_F=\varepsilon^L_F C^{Fm}_{FmLO}\circ \mathrm{P}_F \tag{59}$$

where $(\varepsilon^L_F)^2=+1$ and $\varepsilon^0_F\equiv 1$. Here the Clebsch-Gordan coefficient $C^{\cdot\cdot\cdot}_{\cdot\cdot\cdot}$ of the SO(3) group should be expressed explicitly as the polynomial of degree L in the m operator. We define these polynomials as follows

$$\mathcal{P}^L_F(m)\equiv L!2^{L/2}\{(2L)!\}^{-1/2}c^L_F C^{Fm}_{FmLO}, \tag{60}$$

where c^L_F coefficients which has been defined before (25) can be presented in the form (cf also with (55)),

$$2^L(\hat{F}c^L_F)^2=\frac{(L!)^2}{(2L)!}\frac{(2F+1+L)!}{(2F-L)!}. \tag{61}$$

Using the expression for the Clebsch-Gordan coefficients presented in the monograph by Varschalovich *et al* (1975) or in the book by Jucys and Bandzaitis (1977) one can get

$$\mathscr{P}_F^L(m)=\frac{(L!)^2}{(2L)!}\sum(-)^x\binom{L}{x}^2\frac{(F+m)!(F-m)!}{(F+m+x-L)!(F-m-x)!}$$
$$=m^L-\{\text{terms of the degrees}\leqslant L-2\}. \tag{62}$$

For the few first multipole orders this reads,

$$\mathscr{P}_F^0=\mathrm{i}d, \qquad \mathscr{P}_F^1=m, \qquad \mathscr{P}_F^2=m^2-\tfrac{1}{2}F(F+1)\mathrm{i}d,$$
$$\mathscr{P}_F^3=m^2-\tfrac{1}{5}(3F^2+3F-1)m$$
$$\mathscr{P}_F^4=m^4-\tfrac{1}{7}(6F^2+6F-5)m^2+\tfrac{3}{35}F(F+1)(F-1)(F+2). \tag{63}$$

The equation (58) for e_F^L in terms of the vector s can be written now (inserting (60) into (59) and using (54)):

$$(L!2^{L/2})\cdot\mathscr{F}_F^L(e_F^L)=\varepsilon_F^L[(2L)!]^{1/2}\mathscr{P}_F^L(m)\circ \mathrm{P}_F. \tag{64}$$

In analogy with the recurrent relation (23) we define the rank-L tensor $s^L\in\mathscr{L}_L\subset\overset{L}{\otimes}\mathscr{G}$ in terms of the unit vector $s\in\mathscr{G}$ (see § 10)

$$(s^1\otimes s^B)^L\equiv \mathrm{P}_{AB}^L\circ\{s^A\otimes s^B\}=C_{AOBO}^{LO}s^L(\Rightarrow s^L\cdot s^L=1). \tag{65}$$

Then using essentially (13) and (25) one can prove the following formula

$$[(2L)!]^{1/2}\mathscr{P}_F^L(m)\circ\mathrm{P}_F=2^{L/2}L!\,\mathscr{F}_F^L(s^L). \tag{66}$$

Therefore the final solution of (58) is

$$e_F^L=\varepsilon_F^Ls^L \qquad \text{where} \qquad \varepsilon_F^L=s^L\cdot e_F^L. \tag{67}$$

From (33), and using (64), one can express the degrees of the orientation in terms of the mean values of the so-called moment operators m_F^k, where $m_F\equiv m\circ\mathrm{P}_F$ and $k\in\Delta(F,F)$, namely

$$\hat{L}\langle\mathscr{P}_F^L(m)\circ\mathrm{P}_F\rangle=\frac{(L!)^2}{(2L)!}\left\{\frac{(2F+1+L)!}{(2F-L)!}\right\}^{1/2}g\lambda_F^L\varepsilon_F^L. \tag{68}$$

For the explicit calculations one could use also (55)-(56) and (64).

Equation (68) suggests the use of the case of the axial symmetry of the spin-density operator, instead of the non-negative degrees of the orientation $\{\lambda_F^L\}$, (33), the new degrees

$$\bar{\lambda}_F^L\equiv\varepsilon_F^L\lambda_F^L, \tag{69}$$

which can take also the negative values, $-1\leqslant\bar{\lambda}_F^L\leqslant+1$ for $L\neq 0$.

The cylindrical symmetry has been investigated firstly by Tolhoek and Cox (1953). We also refer to the relevant considerations by Steiger and Fritz (1967). Tolhoek and Cox (1953) noticed that the moment operators $\{m_F^k\}$ for the case of the cylindrical symmetry (and only in this case) span the basis for the spin-density operators. However the polynomials $\mathscr{P}_F^L$ (62) appear to be the more convenient, equivalent basis. The conclusion is well-known: in the case of the cylindrical symmetry the degrees of the orientation $\{\bar{\lambda}_F^L\}$, (68)-(69), describe *completely* the spin-density operator (note that the mapping (37), $\{p_F\}\to\{\lambda_F^0\}$ is invertible). The simple fundamental relation (67) seems never to have been noticed in the full generality. Tolhoek and Cox (1953) defined the degrees of the orientation $\{f^L_F\}$ of the F-states as follows

$$F^L f^L_F \equiv \langle \mathscr{P}^L_F(m) \circ \mathrm{P}_F \rangle. \tag{70}$$

Comparing the Tolhoek and Cox definition with (68) we get rather complicated relation to our degrees of the orientation $\{\bar{\lambda}^L_F\}$. In fact, the normalisation of the f^L_F parameters (70) by Tolhoek and Cox (1953) was rather accidental. The normalisation was suggested by the *special* kind of pure polarisation states, so-called totally oriented states which are defined by the relation

$$\langle m^k_F \rangle = F^k. \tag{71}$$

As the digression relevant for the general multiple expansion (38) in § 4, note that the polynomials $\mathscr{P}^L_F(m)$, (60)-(63) are generating the alternative to $\{\mathscr{P}^L_F\}$ (12), basis of the polarisation operators in End $\mathscr{H}$

$$L!\, F_{i_1 \ldots i_L} \equiv \partial_{i_1} \ldots \partial_{i_L} \mathscr{P}^L_F(m) \tag{72}$$

where $[\partial_i m, \partial_k m] = \mathrm{i}\varepsilon_{ikl}\partial_l m$ and $\partial_i \equiv \partial/\partial s^i$. Here $\{s^i\}$ are the Cartesian components of the vector s (the Cartesian basis in $\mathscr{G}$ is diagonalising symmetric product ϕ, (14)-(15)). One should take into account that $\partial_i \partial_k\, s \cdot s = 2\delta_{ik}$, i.e. that the polynomials $\mathscr{P}^L_F(m)$ in the definition (72) are homogeneous of the degree L in the components of the non-unit vector s, $s \cdot s \neq 1$. For instance

$$2F_{ik} \equiv F_i F_k + F_k F_i - \tfrac{2}{3} F(F+1)\delta_{ik}, \qquad \text{etc.} \tag{73}$$

The above basis (72) has been employed by Bukhvostov and Popov (1964) and by Bukhvostov *et al* (1972). Using the trace formula (19), and also formulae from Varschalovich *et al* (1975) (generalising (32.17) in Werle (1966)), one can find the relation of the basis (72) to (66). Comparing (33) with the corresponding formulae in Bukhvostov *et al* (1972) we have

$$g\hat{F}\hat{L}\lambda^L_F \mathrm{P}^L_F = \xi^{i_1 \ldots i_L}_F F_{i_1 \ldots i_L} \circ p_F \mathrm{P}_F. \tag{74}$$

On the RHS of (74) the factorised form of the spin-density operator (4) is adopted. We will not use the Cartesian basis (72) in the present paper, because for the practical calculations of the angular and polarisation distributions the multipole expansion in the form (38) seems to be the most convenient. It should be stressed however that sometimes the Cartesian basis is more convenient, see Bukhvostov *et al* (1972).

Let us summarise the multipole expansion of the spin-density operator having the axial symmetry,

$$\rho = g \sum \frac{\hat{L}}{\hat{F}} \bar{\lambda}^L_F C^{Fm}_{FmLO} \circ \mathrm{P}_F, \tag{75}$$

where

$$\sum_{F, L \neq 0} (\bar{\lambda}^L_F)^2 \leq 1.$$

The expansion (75) is most convenient for the calculation of the time dependence of the spin-density operator as well as for any kind of the angular and polarisation distributions. However for comparison with previous results we consider more explicitly the first few multipole orders. Inserting (60) into (75) gives

$$\rho = g\sum \hat{L}\frac{(2L)!}{(L!)^2}\left\{\frac{(2F-L)!}{(2F+1+L)!}\right\}^{1/2}\bar{\lambda}^{L}_{F}\mathscr{P}^{L}_{F}(m)\circ \mathrm{P}_F \tag{76}$$

$$\begin{aligned}&=\sum \frac{p_F}{2F+1}\mathrm{P}_F + g\sum \frac{1}{\hat{F}}\left\{\frac{3}{F(F+1)}\right\}^{1/2}\bar{\lambda}^{1}_{F}\mathscr{P}^{1}_{F}(m)\circ \mathrm{P}_F\\
&\quad+g\sum\frac{3}{\hat{F}}\left\{\frac{5}{F(F+1)(2F-1)(2F+3)}\right\}^{1/2}\bar{\lambda}^{2}_{F}\mathscr{P}^{2}_{F}(m)\circ \mathrm{P}_F\\
&\quad+g\sum\frac{5}{\hat{F}}\left\{\frac{7}{F(F+1)(F-1)(F+2)(2F-1)(2F+3)}\right\}^{1/2}\bar{\lambda}^{3}_{F}\mathscr{P}^{3}_{F}(m)\circ \mathrm{P}_F\\
&\quad+\text{terms with } L\geqslant 4.\end{aligned} \tag{77}$$

Using the shorthand notation $\langle\mathscr{P}^{L}_{F}\rangle \equiv \langle\mathscr{P}^{L}_{F}(m)\circ \mathrm{P}_F\rangle$, the first few multipoles in (68), which one often needs in the applications, read

$$\begin{aligned}&3^{1/2}\langle\mathscr{P}^{1}_{F}\rangle = \hat{F}\{F(F+1)\}^{1/2}g\bar{\lambda}^{1}_{F} = 3^{1/2}Ff^{1}_{F}\\
&3(5)^{1/2}\langle\mathscr{P}^{2}_{F}\rangle = \hat{F}\{F(F+1)(2F-1)(2F+3)\}^{1/2}g\bar{\lambda}^{2}_{F} = 3(5)^{1/2}F^2f^{2}_{F}.\end{aligned} \tag{78}$$

It is convenient to calculate the coefficients for higher-order moments from the recurrent relation

$$2\{(2L-1)(2L+1)\}^{1/2}\langle\mathscr{P}^{L}_{F}\rangle\bar{\lambda}^{L-1}_{F} = L\{(2F+1+L)(2F-L+1)\}^{1/2}\langle\mathscr{P}^{L-1}_{F}\rangle\bar{\lambda}^{L}_{F}. \tag{79}$$

It can be stressed that all the formulae in this section do not need the specification of the space $\mathscr{K}$ of the representations of the SO(3) group except that the irreducible decomposition $\mathscr{K} = \oplus\, \mathscr{K}_F$ should *not* contain the multiply irreducible subspaces. This point has been discussed at length at the end of the § 4.

We will end this section with the application of these formulae to the case $\mathscr{K} = [j]$. In this case one can choose $\varepsilon^{1}_{j} = +1$ and moreover one should insert (41) into (76)–(78). Denoting for short $\bar{\lambda}^{L} \equiv \bar{\lambda}^{L}_{F}$, etc, we get explicitly for $j \leqslant \frac{3}{2}$ the following expressions

$$\rho(\tfrac{1}{2}) = \tfrac{1}{2}\{id + 2\lambda^{1}\mathscr{P}^{1}\}, \qquad \text{where} \qquad 2\langle\mathscr{P}^{1}\rangle = \lambda^{1} = f^{1} \geqslant 0, \tag{80}$$

$$\rho(1) = \tfrac{1}{3}\{id + 3^{1/2}\lambda^{1}\mathscr{P}^{1} + 3\bar{\lambda}^{2}\mathscr{P}^{2}\},$$

$$\text{where} \qquad \sqrt{3}\langle\mathscr{P}^{1}\rangle = 2\lambda^{1} = \sqrt{3}\, f^{1}, \qquad 3\langle\mathscr{P}^{2}\rangle = 2\bar{\lambda}^{2} = 3f^{2}, \tag{81}$$

$$\rho(\tfrac{3}{2}) = \tfrac{1}{4}\{id + 2\sqrt{\tfrac{3}{5}}\,\lambda^{1}\mathscr{P}^{1} + \sqrt{3}\,\bar{\lambda}^{2}\mathscr{P}^{2} + 2\sqrt{\tfrac{5}{3}}\,\bar{\lambda}^{3}\mathscr{P}^{3}\},$$

$$\begin{aligned}\text{where} \qquad &2\langle\mathscr{P}^{1}\rangle = 15^{1/2}\lambda^{1} = 3f^{1}, \qquad 4\langle\mathscr{P}^{2}\rangle = 4(3)^{1/2}\bar{\lambda}^{2} = 9f^{2},\\
&8(5)^{1/2}\langle\mathscr{P}^{3}\rangle = 12(3)^{1/2}\bar{\lambda}^{3} = 27(5)^{1/2}f^{3}.\end{aligned} \tag{82}$$

The above spin-density operators $\rho(j)$ describe the completely polarised pure states iff $\mathrm{Tr}[\rho(j)]^2 = 1 \Leftrightarrow \Sigma_{L\neq 0}(\bar{\lambda}^{L})^2 = 1$. The multipole expansion for $\rho(1)$, (81), has been derived previously by Bukhvostov *et al* (1972). As was noticed in this unpublished report the spin-density operator $\rho(1)$ describes the totally oriented Tolhoek-Cox state (71) iff $\lambda^{1} = \frac{1}{2}\sqrt{3}$ and $\bar{\lambda}^{2} = \frac{1}{2}$.

7. Multipole expansion in terms of the individual spin operators

The basis in End[j] of the polarisation operators discussed in § 3 will induce, through the Clebsch-Gordan decomposition, the new basis of the SO(3)-irreducible tensor operators in $\mathrm{End}([j_1]\otimes[j_2])$,

$$\mathcal{R}^L_{fe} \equiv (\mathcal{P}^f_{j_1}\otimes\mathcal{P}^e_{j_2})^L \equiv \mathcal{P}^f_{j_1}\otimes\mathcal{P}^e_{j_2}) \circ \mathrm{P}^L_{fe}. \tag{83}$$

This set of operators is relevant for the description of the spin-density operator in terms of the polarisations of the subsystems. Using the Racah algebra (Varschalovich *et al* 1975, Jucys and Bandzaitis 1977) one can easily derive the simple relations between both these complete bases (12) and (83)

$$\mathcal{P}^L_{FE} = \frac{\hat{F}\hat{F}\hat{E}}{\hat{L}\hat{j}_1\hat{j}_2}\sum (\hat{f}\hat{e})^2 \begin{Bmatrix} j_1 & j_2 & F \\ j_1 & j_2 & E \\ f & e & L \end{Bmatrix} \mathcal{R}^L_{fe} \tag{84}$$

$$\mathcal{R}^L_{fe} = \hat{L}\hat{j}_1\hat{j}_2 \sum \hat{E} \begin{Bmatrix} j_1 & j_2 & F \\ j_1 & j_2 & E \\ f & e & L \end{Bmatrix} \mathcal{P}^L_{FE}. \tag{85}$$

The spin operator in $[j]$ (see (21)) is given by

$$\mathcal{S}_j \equiv c^1_j \mathcal{P}^1_j \qquad : \mathcal{G} \to \mathrm{End}[j]. \tag{86}$$

Then the spin operators of the subsystems are

$$\mathcal{J}_1 \equiv \mathcal{S}_{j_1} \otimes id = c^1_{j_1}\mathcal{R}^1_{10} \qquad : \mathcal{G} \to \mathrm{End}([j_1]\otimes[j_2]) \tag{87}$$

$$\mathcal{J}_2 \equiv id \otimes \mathcal{S}_{j_2} = c^1_{j_2}\mathcal{R}^1_{01} \qquad : \mathcal{G} \to \mathrm{End}([j_1]\otimes[j_2]). \tag{88}$$

Using (85) one can show the obvious equality

$$\mathcal{F} = \mathcal{J}_1 + \mathcal{J}_2, \tag{89}$$

when the LHS is defined by (21). Similarly to the rank-L total spin operator $\mathcal{F}^L$ (see § 3) we define rank-L spin operators of the system

$$\mathcal{S}^L_{fe} \equiv c^f_{j_1} c^e_{j_2} R^L_{fe} = (\mathcal{S}^f_{j_1}\otimes\mathcal{S}^e_{j_2}) s^f_{j_1}\otimes s^e_{j_2} \circ \Phi \circ P^L. \tag{90}$$

It should be evident that the $\mathcal{S}^L_{fe}$ tensor operators are built up entirely from the individual spin operators (86)-(88) by means of (23). Inserting (84) into (26) we obtain the multipole expansion of the arbitrary operator $\rho \in \mathrm{End}([j_1]\otimes[j_2])$ in terms of the individual spin operators $\{\mathcal{S}^L_{fe}\}$,

$$\rho = \sum_{feL} (-)^{f+e+L}\left(\frac{\hat{f}\hat{e}}{\hat{j}_1\hat{j}_2}\right)^2 \langle \mathcal{R}^L_{fe}\rangle \cdot \mathcal{R}^L_{fe} \tag{91}$$

where the summation is for $\forall f \in \Delta(j_1 j_1)$, $\forall e \in \Delta(j_2 j_2)$. The conditions (29) and (85), give

$$\langle \mathcal{R}^L_{fe}\rangle = 0 \qquad \text{for } f+e+L = \text{odd}. \tag{92}$$

For the calculations of the angular and the polarisation distributions with the spin-density operator fulfilling condition (28), the most convenient seem to be (38) and (75) rather than (91). At the same time the new parameters $\langle \mathcal{S}^L_{fe}\rangle$ can be more interesting for theoretical considerations, e.g. for investigations of the correlations and models of the depolarisations, as well as from the experimental point of view. However, they are no longer independent which follows from (29) and (84). In fact we have the identities

$$\sum_{f,e} (\hat{f}\hat{e})^2 \begin{Bmatrix} j_1 & j_2 & F \\ j_1 & j_2 & E \\ f & e & L \end{Bmatrix} \langle \mathcal{R}^L_{fe}\rangle = 0 \qquad \text{for } F \neq E. \tag{93}$$

These linear relations play the role of the constraints which assure that the compositions of the mappings (94) and (95) below are the identity mappings.

Inserting (28)-(29) and (33) into (84) and (85) we get

$$\hat{j}_1\hat{j}_2 g\lambda^L_F e^L_F = \hat{F}^2 \sum (\hat{f}\hat{e})^2 \begin{Bmatrix} j_1 & j_2 & F \\ j_1 & j_2 & F \\ f & e & L \end{Bmatrix} \langle \mathcal{R}^L_{fe} \rangle, \tag{94}$$

$$\langle \mathcal{R}^L_{fe} \rangle = \hat{j}_1\hat{j}_2 g \sum \hat{F}^2 \begin{Bmatrix} j_1 & j_2 & F \\ j_1 & j_2 & F \\ f & e & L \end{Bmatrix} \lambda^L_F e^L_F. \tag{95}$$

From (85) follows the relation important from the experimental point of view,

$$\mathrm{P}_F \circ \mathcal{R}^L_{fe} \circ \mathrm{P}_F = \hat{L}\hat{F}\hat{j}_1\hat{j}_2 \begin{Bmatrix} j_1 & j_2 & F \\ j_1 & j_2 & F \\ f & e & L \end{Bmatrix} \mathcal{P}^L_F. \tag{96}$$

Using (86)-(90) one can express the mean values of the rank-L total spin operator in F-state through the same rank spin operator of the system (90) in the F-state

$$\langle \mathrm{P}_F \circ \mathcal{S}^L_{fe} \circ \mathrm{P}_F \rangle = c^f_{j_1} c^e_{j_2} (c^L_F)^{-1} \hat{L}\hat{F}\hat{j}_1\hat{j}_2 \begin{Bmatrix} j_1 & j_2 & F \\ j_1 & j_2 & F \\ f & e & L \end{Bmatrix} \langle \mathcal{F}^L \circ \mathrm{P}_F \rangle. \tag{97}$$

The above formula gives the alternative devices for the experimental measurement of the populations $\{p_F\}$ and of the degrees of the orientation $\{\lambda^L_F\}$. One needs to recall how they are related to the mean values of the total spin operator (33):

$$\hat{L}(c^L_F)^{-1} \langle \boldsymbol{F}^L \circ \mathrm{P}_F \rangle \equiv g\hat{F}\lambda^L_F \boldsymbol{e}^L_F. \tag{98}$$

As

$$\mathcal{S}^0_{ff} = (-)^f \hat{f}^{-1} \mathcal{S}^f_{j_1} \cdot \mathcal{S}^f_{j_2} \in \mathrm{End}\, \mathcal{H}, \tag{99}$$

therefore (97) for $L=0$ gives

$$\langle \mathcal{S}^f_{j_1} \cdot \mathcal{S}^f_{j_2} \circ \mathrm{P}_F \rangle = (-)^{j_1+j_2+F} c^f_{j_1} c^f_{j_2} \frac{\hat{j}_1\hat{j}_2}{2f+1} \begin{Bmatrix} j_1 & j_2 & F \\ j_2 & j_1 & f \end{Bmatrix} \langle \mathrm{P}_F \rangle, \tag{100}$$

and in particular for $f=1$,

$$\langle \mathcal{S}_{j_1} \cdot \mathcal{S}_{j_2} \circ \mathrm{P}_F \rangle = \tfrac{1}{6}\{F(F+1) - j_1(j_1+1) - j_2(j_2+1)\} p_F. \tag{101}$$

For $L=1$ we list the two most important relations which follow from (97):

$$\langle \mathrm{P}_F \circ \mathcal{J}_1 \circ \mathrm{P}_F \rangle = \tfrac{1}{2}\left(1 + \frac{j_1(j_1+1) - j_2(j_2+1)}{F(F+1)}\right) \langle \mathcal{F} \circ \mathrm{P}_F \rangle \tag{102}$$

$$\langle \mathrm{P}_F \circ \mathcal{J}_2 \circ \mathrm{P}_F \rangle = \tfrac{1}{2}\left(1 + \frac{j_2(j_2+1) - j_1(j_1+1)}{F(F+1)}\right) \langle \mathcal{F} \circ \mathrm{P}_F \rangle. \tag{103}$$

We would like to stress that all consequences of (84)-(85) are quite general and are completely model independent. In particular the above formulae tell us how the mean value of the total spin operator in the F-state is related to the mean value of the spin operators for each of the subsystem. Obviously (84)-(85) contain much more useful information than we explore in the present paper. We note however that (84) or (94) for $L=0$ gives the important model-independent expression for the non-statistical populations

$$\mathrm{P}_F = \overset{\text{stat}}{\mathrm{P}}_F + (-)^{j_1+j_2+F}\frac{\hat{F}^2}{\hat{j}_1\hat{j}_2}\sum_{f\neq 0}(-)^f\hat{f}^3\begin{Bmatrix} j_1 & j_2 & F \\ j_2 & j_1 & f \end{Bmatrix}\langle \mathscr{R}^0_{ff}\rangle, \tag{104}$$

because

$$(85)\Rightarrow \mathrm{Tr}\,\rho = 1 \Leftrightarrow \langle \mathscr{R}^0_{00}\rangle = 1. \tag{105}$$

Here $\overset{\text{stat}}{p}_F$ is given by (39) and { } in (100) and (104) denote the Racah coefficient (Varschalovich *et al* 1975, Jucys and Bandzaitis 1977). Inserting (104) into (32) gives

$$g^2 = (\overset{\text{stat}}{g})^2 - \sum \frac{(p_F - \overset{\text{stat}}{p}_F)^2}{2F+1}$$

with

$$\sum \frac{(p_F - \overset{\text{stat}}{p}_F)^2}{2F+1} = (\hat{j}_1\hat{j}_2)^{-1}\sum_{f\neq 0}\hat{f}^2\langle \mathscr{R}^0_{ff}\rangle^2. \tag{106}$$

The rest of this section will be devoted to the case $j_1 = \frac{1}{2}$ and $j_2 = j$ *arbitrary*. The appropriate expressions for the Fano coefficients are taken from Varschalovich *et al* (1975) or Jucys and Bandzaitis (1977, formulae (32.15*a*) and (32.17)). In the appendix we collect together a few formulae for the Fano, Racah and Clebsch–Gordan coefficients which we need frequently.

First of all (93) are reduced to the relation

$$\begin{aligned}\hat{L}^2\{L(L+1)\}^{1/2}\langle \mathscr{R}^L_{0L}\rangle &= \{3L(2L+3)(2j-L)(2j+L+2)\}^{1/2}\langle \mathscr{R}^L_{1L+1}\rangle \\ &\quad + \{3(L+1)(2L-1)(2j-L+1)(2j+L+1)\}^{1/2}\langle \mathscr{R}^L_{1L-1}\rangle. \end{aligned}\tag{107}$$

Equation (94) can be presented separately for each hyperfine level $F = j \pm \frac{1}{2}$:

$$\begin{aligned} 2\hat{L}\hat{j}^3 g\lambda^L_+ e^L_+ &= \hat{L}^2\{(2j+L+2)(2j-L+1)\}^{1/2}\langle \mathscr{R}^L_{0L}\rangle \\ &\quad + \{3L(2L-1)(2j+L+1)(2j+L+2)\}^{1/2}\langle \mathscr{R}^L_{1L-1}\rangle \\ &\quad - \{3(L+1)(2L+3)(2j-L)(2j-L+1)\}^{1/2}\langle \mathscr{R}^L_{1L+1}\rangle, \\ 2\hat{L}\hat{j}^3 g\lambda^L_- e^L_- &= \hat{L}^2\{(2j+L+1)(2j-L)\}^{1/2}\langle \mathscr{R}^L_{0L}\rangle \\ &\quad - \{3L(2L-1)(2j-L)(2j-L+1)\}^{1/2}\langle \mathscr{R}^L_{1L-1}\rangle \\ &\quad + \{3(L+1)(2L+3)(2j+L+1)(2j+L+2)\}^{1/2}\langle \mathscr{R}^L_{1L+1}\rangle. \end{aligned}\tag{108}$$

In particular from (108) for $L=0$ we obtain the general expression for the non-statistical populations

$$p_\pm = \overset{\text{stat}}{p}_\pm \mp 3\frac{[j(j+1)]^{1/2}}{2j+1}\langle \mathscr{R}^0_{11}\rangle, \tag{109}$$

and

$$(\overset{\text{stat}}{g})^2 - g^2 = \frac{2j+1}{2j(j+1)}(p - \overset{\text{stat}}{p})^2 = \tfrac{9}{2}\hat{j}^{-2}\langle \mathscr{R}^0_{11}\rangle^2. \tag{110}$$

Expansion (95) can now be presented as the set of the three formulae:

$$\hat{j}\hat{L}\langle \mathcal{R}^{L}_{0L}\rangle = \{(2j+L+2)(2j-L+1)\}^{1/2} g\lambda^{L}_{+}e^{L}_{+} + \{(2j+L+1)(2j-L)\}^{1/2} g\lambda^{L}_{-}e^{L}_{-},$$

$$3^{1/2}\hat{j}\hat{L}\langle \mathcal{R}^{L}_{1L-1}\rangle = g\left(\frac{1}{2L-1}\right)^{1/2}\{[(2j+L+1)(2j+L+2)]^{1/2}\lambda^{L}_{+}e^{L}_{+}$$
$$-[(2j-L)(2j-L+1)]^{1/2}\lambda^{L}_{-}e^{L}_{-}\}, \qquad (111)$$
$$3^{1/2}\hat{j}\hat{L}\langle \mathcal{R}^{L}_{1L+1}\rangle = g\left(\frac{L+1}{2L+3}\right)^{1/2}\{-[(2j-L)(2j-L+1)]^{1/2}\lambda^{L}_{+}e^{L}_{+}$$
$$+[(2j+L+1)(2j+L+2)]^{1/2}\lambda^{L}_{-}e^{L}_{-}\}.$$

Finally the most important consequences of the identity (97) seems to be the following one

$$\langle \mathrm{P}_{+}\circ \mathscr{j}^{L}_{2}\circ \mathrm{P}_{+}\rangle = \left(1-\frac{L}{2j+1}\right)\langle \mathscr{F}^{L}\circ \mathrm{P}_{+}\rangle, \qquad \langle \mathrm{P}_{-}\circ \mathscr{j}^{L}_{2}\circ \mathrm{P}_{-}\rangle = \left(1+\frac{L}{2j+1}\right)\langle \mathscr{F}^{L}\circ \mathrm{P}_{-}\rangle, \qquad (112)$$

and

$$\langle \mathrm{P}_{\pm}\circ \mathscr{S}^{L}_{1L-1}\circ \mathrm{P}_{\pm}\rangle = \pm\frac{1}{2j+1}\langle \mathscr{F}^{L}\circ \mathrm{P}_{\pm}\rangle \qquad (113)$$

should be useful for phenomenological analysis. In particular (96)-(97) and their consequences (112)-(113) can be used in the multipole expansion of the spin-density operator. Let us note that (113) for $L=1$ leads to

$$\langle \mathrm{P}_{\pm}\circ \mathscr{j}_{1}\circ \mathrm{P}_{\pm}\rangle = \pm\frac{1}{2j+1}\langle \mathscr{F}\circ \mathrm{P}_{\pm}\rangle, \qquad (114)$$

which has been frequently used by Bukhvostov and Popov (1967) and (1970).

The same remark concern also to the above general formulae (104), (109), for the non-statistical populations: they are completely *model independent.* The only parameter $\langle \mathcal{R}^{0}_{11}\rangle$ which is responsible for the non-statistical populations for the case $\rho(\frac{1}{2}\otimes j)$ can and should be determined experimentally. We have

$$\langle \mathcal{R}^{0}_{11}\rangle \overset{(90)}{=} \langle c^{1}_{j_1}c^{1}_{j_2}\rangle^{-1}\langle \mathscr{S}^{0}_{11}\rangle \overset{(22)\text{ and }(100)}{=} -\{3j_1(j_1+1)j_2(j_2+1)\}^{-1/2}\langle \mathscr{S}_{j_1}\cdot \mathscr{S}_{j_2}\rangle$$
$$= -\tfrac{2}{3}\{j(j+1)\}^{-1/2}\langle \mathscr{S}_{1/2}\cdot \mathscr{S}_{j}\rangle. \qquad (115)$$

For each concrete model of the atomic depolarisation (Djrbashyan 1959, Shmushkevich 1959, Bukhvostov and Popov 1964, Bukhvostov 1966, 1969, Bukhvostov *et al* 1972) this parameter can be calculated as a function of the initial polarisation of the free system. Such calculations of the $\langle \mathscr{S}_{1/2}\cdot \mathscr{S}_{j}\rangle$ parameter were first performed by Bukhvostov and Popov (1964) for the case $\rho(\frac{1}{2}\otimes\frac{1}{2})$ and then for $j=1$ by Bukhvostov *et al* (1972) in the simplest model of the depolarisation which we consider in §§ 8 and 9. Hambro and Mukhopadyay (1975) calculated $\langle \mathscr{S}_{1/2}\cdot \mathscr{S}_{j}\rangle$ for the arbitrary nuclear target spin j in the same model. We will discuss these results in § 9 which is devoted to the simplest model of the non-statistical hyperfine populations. The experimental determination of the $\langle \mathscr{S}_{1/2}\cdot \mathscr{S}_{j}\rangle$ parameter in (109) with (115), should be very important for the verification of the theoretical models of the process of the formation of the miu-mesic atom (which is referred to for short as the depolarisation process).

The parameters a_{efL} introduced by Bukhvostov and Popov (1967, 1970) (see also Bukhvostov *et al* 1972, Oziewicz 1977) are simply related to $\langle \mathcal{R}^L_{fe} \rangle$. Comparing the decomposition (91)-(92) with for instance equations (8)-(9) of Bukhvostov *et al* (1972) we get

$$a_{efL} \equiv (-)^f \hat{e}\hat{f}^2 \hat{L} \langle \mathcal{R}^L_{fe} \rangle. \tag{116}$$

We also note that the $\boldsymbol{q}^L$ statistical tensors used in Oziewicz (1977) are related to the degrees of the orientation of the present paper as follows

$$\boldsymbol{q}^L \equiv \boldsymbol{q}^L_F \equiv g\hat{F}\lambda^L_F \boldsymbol{e}^L_F \qquad \text{with } q_F \equiv \boldsymbol{q}^1_F \cdot \boldsymbol{s}, \tag{117}$$

where s is the direction of the axial symmetry (unit vector of the muon polarisation). The parameters Q_{efL} (Oziewicz 1977) are obviously F-independent (the index F should be removed from equations (5)-(6) (Oziewicz 1977)). Formula (7) in Oziewicz (1977) is nothing other than another form of (111), i.e. summation over F is understood on the RHS. Note that

$$2\boldsymbol{s} \cdot \langle \boldsymbol{j}_1 \rangle = Q_{011}. \tag{118}$$

The identity (114) for the case of the axial symmetry of the spin-density operator with the direction of the symmetry described by the unit vector s, allows us to define the Bukhvostov and Popov parameters

$$p_F \lambda^{\text{BP}}_F \equiv 2\boldsymbol{s} \cdot \langle \mathrm{P}_F \circ \boldsymbol{j}_1 \circ \mathrm{P}_F \rangle. \tag{119}$$

If $[\rho, \mathrm{P}_F] = 0 \ \forall F$, then (119) is identical to definition (9) in Bukhvostov and Popov (1970), and should be compared with (53).

8. Depolarisation: The simplest model

The pioneering work on the theory of the atomic depolarisation is due to Djrbashyan (1959) and Shmushkevich (1959). The theory has been subsequently developed by Bukhvostov and Popov (1964) and by Bukhvostov (1966, 1969). In this section and § 9 we generalise the simplest model of the depolarisation considered by Bukhvostov *et al* (1972) for $\mathcal{H} = [\frac{1}{2}] \otimes [1]$ and then by Hambro and Mukhopadhyay (1975) for $\mathcal{H} = [\frac{1}{2}] \otimes [j]$ to the general case $\mathcal{H} = [j_1] \otimes [j_2]$. In this simplest model it is assumed that the depolarisation is due to the spin-spin (i.e. hyperfine) interaction of the magnetic moments on the K-shell only. This leads to the hyperfine splitting of the K-shell. However, a more reaslistic theory of the depolarisation should also take into account the fine and hyperfine splitting of the excited atomic shells (see the papers by Bukhvostov *et al* previously cited).

The initial, not correlated system (the free case) is described by the tensor product of the spin-density operators of the individual subsystems,

$$\rho(j_1) \otimes \rho(j_2) \in (\mathrm{End}[j_1]) \otimes (\mathrm{End}[j_2)] \subset \mathrm{End}([j_1] \otimes [j_2]). \tag{120}$$

The arbitrary operator $\mathcal{O} \in \mathrm{End}\, \mathcal{H}$ can be decomposed into the two parts:

$$\mathcal{O} = \sum \mathrm{P}_F \circ \mathcal{O} \circ \mathrm{P}_F + \sum_{F \neq E} \mathrm{P}_F \circ \mathcal{O} \circ \mathrm{P}_E, \tag{121}$$

where the first term is the polarisation operator, or *the polarised part* of $\mathcal{O}$. The polarised part and, in particular, every polarisation operator is the integral of the motion with respect to the spin-spin interaction with the Hamiltonian operator (cf (90) and (99)),

$$H \equiv \sum a_f \mathcal{S}^f_{j_1} \cdot \mathcal{S}^f_{j_2}, \tag{122}$$

where the summation is for $0 < f \leqslant \min(2j_1, 2j_2)$. Here $\{a_f\}$ are arbitrary coupling constants. The above Hamiltonian operator is commuting with the all polarisation operators in $\mathcal{K}$. The time averaged second part in the decomposition (121) vanishes. The splitting of the K-shell does depend on the coupling constants $\{a_f\}$. However, any kind of hyperfine (HF) interaction (122) leads to the same model of the spin-density operator of the atom

$$\rho(j_1) \otimes \rho(j_2) \xrightarrow{\text{HF}} \rho(j_1 \otimes j_2) = \sum \mathrm{P}_F \circ \rho(j_1) \otimes \rho(j_2) \circ \mathrm{P}_F. \tag{123}$$

The spin-density operator in the above model can be represented in the form (38) with the degrees of the orientation of the atom expressed in terms of the degrees of the orientations of the individual free subsystems exactly according to (94). To see this we will denote $\langle\ \rangle_0$ the mean value of the corresponding operator with respect to the product (120) of the individual spin-operators, and by $\langle\ \rangle$ the mean value with respect to the model spin-density operator (123). Then

$$\langle \mathcal{P}^L_{FE} \rangle = \delta_{FE} \langle \mathcal{P}^L_F \rangle_0 \qquad \text{and} \qquad \langle \mathcal{R}^L_{fe} \rangle_0 \neq \langle \mathcal{R}^L_{fe} \rangle. \tag{124}$$

Therefore calculating firstly the $\langle \mathcal{P}^L_F \rangle_0$ according to (85) and *afterwards* using (123) we obtain (94) with $\langle \mathcal{R}^L_{fe} \rangle_0$ parameters instead of $\langle \mathcal{R}^L_{fe} \rangle$ under the sign of the summation on the RHS. It is very important to realise that for $\langle \mathcal{R}^L_{fe} \rangle_0$ none of the relations (92), (93) or (95) hold. Simply, all Fano parameters of the subsystems are independent. Contrary to the constraints (93), (94) with $\langle \mathcal{R}^L_{fe} \rangle_0$ parameters on the RHS, now imply the relations between the degrees of the orientation of the atom $\{\lambda^L_F\}$, they are no longer independent. These relations serve as an experimental test of the above model of the depolarisation (123).

Comparing (120) with the general multipole expansion (91) and using (83) we get the well known formula (Devons and Goldfarb 1957, Goldfarb and Bromley 1962, equation (5.1)),

$$\langle R^L_{fe} \rangle_0 = (\langle P^f_{j_1} \rangle \otimes \langle P^e_{j_2} \rangle), \tag{125}$$

where

$$\langle \mathcal{P}^f_j \rangle \equiv \mathrm{Tr}\{\rho(j) \circ \mathcal{P}^f_j\} = \{2j\}^{1/2} \hat{f}^{-1} \lambda^f_j e^f_j, \tag{126}$$

are the Fano polarisation parameters of the individual free subsystem (cf with (27) and (42)).

The rest of this section will be devoted to the important case of the statistical hyperfine populations $p_F = \overset{\text{stat}}{p_F}$ in the above model. In § 9 we will collect together the formulae and discuss the several examples relevant for the general case $p_F \neq \overset{\text{stat}}{p_F}$ (see (104)) of the non-statistical hyperfine populations in the same model.

Suppose that the second subsystem is completely unpolarised

$$\rho(j_2) = \mathrm{i}d/(2j_2 + 1),$$

then

$$\langle \mathcal{R}^L_{fe} \rangle_0 = \delta_{e,0} \delta_{f,L} \langle \mathcal{P}^L_{j_1} \rangle \qquad \text{and} \qquad (104) \Rightarrow p_F = \overset{\text{stat}}{p_F}. \tag{127}$$

It is now a simple matter to express the mean value of the rank-L total spin operator in any hyperfine state F through the mean value of the rank-L spin operator of the first subsystem in the initial free state. To do this one should recall (21) and (90), and use (63) and (85). The result is

$$\langle \mathscr{F}_F^L \rangle = (-)^{j_1+j_2+F+L} \left(\frac{\hat{F}}{\hat{j}_2} \right)^2 \left\{ \frac{(2F+1+L)!(2j_1-L)!}{(2F-L)!(2j_1+1+L)!} \right\}^{1/2} \begin{Bmatrix} F & j_1 & j_2 \\ j_1 & F & L \end{Bmatrix} \langle \mathscr{S}_{j_1}^L \rangle. \tag{128}$$

For $L = 1$, using the expression for the Racah coefficient given in the appendix, we get

$$\langle \mathscr{F}_F \rangle = \tfrac{1}{2} \left(\frac{\hat{F}}{\hat{j}_1 \hat{j}_2} \right)^2 \left(1 + \frac{F(F+1) - j_2(j_2+1)}{j_1(j_1+1)} \right) \langle \mathscr{S}_{j_1} \rangle, \tag{129}$$

which, for $j_1 = \frac{1}{2}$ and $j_2 = j$, reduces to

$$\langle \mathscr{F} \circ P_F \rangle = \frac{2F+1}{3(2j+1)} [1 + (F-j)(2j+1)] \langle \mathscr{S}_{1/2} \rangle. \tag{130}$$

The last formula in the full line reads

$$\langle \mathscr{F}_+ \rangle = \frac{(j+1)(2j+3)}{3(2j+1)} \langle \mathscr{S}_{1/2} \rangle, \qquad \langle \mathscr{F}_- \rangle = \frac{j(1-2j)}{3(2j+1)} \langle \mathscr{S}_{1/2} \rangle. \tag{131}$$

Obviously (130)-(131) imply automatically the axial symmetry of the spin-density operator.

All relations (128)-(131) can be experimentally tested for verification of the *validity* of the model (123) in the reactions with the one (second) subsystem unpolarised (127).

Inserting into (128) the degrees of orientation according to expressions (33)-(36) with (43) for $\overset{\text{stat}}{g}$ and according to (42) or (126) we obtain

$$\lambda_F^L = \frac{(2F+1)(j_1)^{1/2}}{\hat{j}_2(j_1+j_2+2j_1j_2)^{1/2}} \left| \begin{Bmatrix} F & j_1 & j_2 \\ j_1 & F & L \end{Bmatrix} \right| \lambda_{j_1}^L, \tag{132}$$

with

$$\boldsymbol{e}_F^L \cdot \boldsymbol{e}_{j_1}^L = \operatorname{sign}\left\{ (-)^{j_1+j_2+F+L} \begin{Bmatrix} F & j_1 & j_2 \\ j_1 & F & L \end{Bmatrix} \right\}. \tag{133}$$

(We recall that the degrees of the orientation have been defined as the non-negative numbers, see (33)). For $L = 1$ (132)-(133), or (129) yields

$$\lambda_F = \frac{\hat{F}}{2\hat{j}_1\hat{j}_2} \cdot \frac{|F(F+1) + j_1(j_1+1) - j_2(j_2+1)|}{\{(j_1+j_2+2j_1j_2)(j_1+1)F(F+1)\}^{1/2}} \lambda_{j_1}, \tag{134}$$

with

$$\boldsymbol{e}_F \cdot \boldsymbol{e}_{j_1} = \operatorname{sign}\{F(F+1) + j_1(j_1+1) - j_2(j_2+1)\}. \tag{135}$$

Formulae (128)-(135) generalise for arbitrary spins j_1 and j_2, what has been considered previously by Bukhvostov and Popov (1967, 1970) for the case $j_1 = \frac{1}{2}$ and $j_2 = j$. This case corresponds, for example, to the initial state of the miu-mesic atom in the capture of the polarised spin-$\frac{1}{2}$ muons by *unpolarised* spin-j target nuclei. Also this is the particular (model-dependent) case of the general situation which has been considered in § 5, when the atom $\rho(\frac{1}{2} \otimes j)$ has vector polarisation only. As $0 \leq \lambda_{j_1}^L \leq 1$ (for $L \neq 0$), then the degrees of the orientation of the atom $\{\lambda_F^L\}$ are bound from above, *in the model of the depolarisation* considered here, according to (132) and (134).

For the case considered previously by Bukhvostov and Popov, $j_1=\frac{1}{2}$ and $j_2=j$, we put $\lambda\equiv\lambda_{j_1}$ in (134) and obtain

$$\lambda_F=\frac{\hat{F}}{2^{1/2}\hat{j}}\cdot\frac{|1+(F-j)(2j+1)|}{\{3(4j+1)F(F+1)\}^{1/2}}\lambda, \tag{136}$$

$$e_F\cdot s=\operatorname{sign}\{1+(F-j)(2j+1)\}, \tag{137}$$

where $s\equiv e_{j_1}$ describes the direction of the axial symmetry. More explicitly (136) can be written as follows

$$\lambda_+=\frac{\lambda}{2j+1}\left\{\frac{(j+1)(2j+3)}{3(4j+1)}\right\}^{1/2}, \qquad \lambda_-=\frac{\lambda}{2j+1}\left\{\frac{j(2j-1)}{3(4j+1)}\right\}^{1/2} \tag{138}$$

The degree of the *total* orientation of the system $\rho(\frac{1}{2}\otimes j)$ with the vector polarisation only

$$\lambda^{\text{tot}}\equiv\{(\lambda_+)^2+(\lambda_-)^2\}^{1/2}\leqslant 1, \tag{139}$$

and can be presented in the form

$$\lambda^{\text{tot}}=d_j\lambda, \tag{140}$$

where λ is the degree of the initial polarisation of the spin-$\frac{1}{2}$ subsystem $(0\leqslant\lambda\leqslant 1)$. For the model of depolarisation considered here, inserting (138) into (139), we obtain

$$d_j=\frac{1}{2j+1}\left\{\frac{4j^2+4j+3}{3(4j+1)}\right\}^{1/2}\xrightarrow[j\to\infty]{}\left(\frac{1}{12j}\right)^{1/2}. \tag{141}$$

In order to obtain the bounds for the Bukhovostov and Popov parameters (see (53) and (119)) one should insert (136)-(138) into (53) with

$$p_F=\overset{\text{stat}}{p_F}\Rightarrow g=\overset{\text{stat}}{g}=\left\{\frac{4j+1}{4j+2}\right\}^{1/2} \tag{142}$$

(this follows from (127) and (45)). Also taking into account that $(67)\Rightarrow e_F=\varepsilon_F s$ we obtain†

$$\lambda_F^{\text{BP}}=\frac{4F-2j+1}{3(2j+1)}\lambda, \tag{143}$$

i.e.

$$|\lambda_F^{\text{BP}}|\leqslant\frac{4F-2j+1}{3(2j+1)}. \tag{144}$$

We would like to stress that the bound (144) follows from the model of the depolarisation (123).

For a proper understanding of formulae (132)-(144) the physical meaning of the degrees of the orientations $\{\lambda_F^L\}$ and $\{\lambda_{j_1}^L\}$ should be made clear. This can be read off from the expressions (36), (42), (55)-(56) and the model independent relations derived in the § 7 (like (97), (103), (112)-(114)). (The model-dependent relations (128)-(144) should be *tested* experimentally and cannot be used as the definitions!). The definitions

† The inequality after formula (11) in Bukhvostov and Popov (1967) contains an unimportant misprint, related probably to the redefinition of the Tolhoek and Cox (1953) parameters, cf (119) with the definitions (145).

and the identities derived in the §§ 4, 5 and 7 show how the degrees of the orientation are related to the mean values of the appropriate spin operators. We shall recall these definitions once more for $L=1$ and for the *statistical* hyperfine populations (127).

$$(80) \Rightarrow 2\boldsymbol{s}\cdot\langle \boldsymbol{S}_{1/2}\rangle = \lambda = f \geqslant 0.$$

From (52) for $g = \overset{\text{stat}}{g}$ (142) and using (114)

$$\begin{aligned} 2\boldsymbol{s}\cdot\langle \boldsymbol{\jmath}_1 \circ P_+\rangle &= \frac{1}{2j+1}\{\tfrac{1}{3}(j+1)(2j+3)(4j+1)\}^{1/2}\lambda_+ = f_+ \\ 2\boldsymbol{s}\cdot\langle \boldsymbol{\jmath}_1 \circ P_-\rangle &= \frac{1}{2j+1}\{\tfrac{1}{3}j(2j-1)(4j+1)\}^{1/2}\lambda_- = \frac{1-2j}{1+2j}f_-. \end{aligned} \tag{145}$$

Here generally, according to (33)-(34), $0 \leqslant \lambda_F \leqslant 1$, and for the Tolhoek-Cox parameters we obtain $f_+ \geqslant 0$ and $f_- \leqslant 0$. Definitions (145) are obviously model independent.

We shall now calculate parameter (118) starting from (145) and using the model expressions (138). The result is

$$2\boldsymbol{s}\cdot\langle \boldsymbol{\jmath}_1\rangle = (4j+1)d_j^2\lambda (= Q_{011} \equiv -\tfrac{1}{3}A_{011}), \tag{146}$$

where d_j is given above (141).

9. The spin-density operator for initially both subsystems oriented in the simplest model of the depolarisation: non-statistical hyperfine populations

The case when for the *both* initial free subsystems $\rho(j) \neq \mathrm{i}d/(2j+1)$ has been firstly considered by Bukhvostov and Popov (1964) for $\mathcal{K} = [\frac{1}{2}]\otimes[\frac{1}{2}]$, then by Bukhvostov *et al* (1972) for $\mathcal{K} = [\frac{1}{2}]\otimes[1]$. Hambro and Mukhopadhyay (1975) considered the case $\mathcal{K} = [\frac{1}{2}]\otimes[j]$ with the arbitrary target spin j, however, contrary to previous papers, they restricted investigations to the resulting populations (in the model of the depolarisation which we described in § 8). In §§ 7 and 8 we generalised these results to the case $\mathcal{K} = [j_1]\otimes[j_2]$ (see the formulae (94) and (104)). We gave the full description of the resulting spin-density operator (123) i.e. the formulae for the populations *and* for the resulting degrees of the orientation of the atom in terms of the degrees of the orientations of the initial free subsystems.

In this section we would like to discuss in more detail the case $\mathcal{K} = [\frac{1}{2}]\otimes[j]$ i.e. (108)-(110) in the model (123).

Firstly we shall recall the model-independent formulae for the populations. Inserting (115) into (109)-(110) we have

$$p_F = \overset{\text{stat}}{p_F} + [4(F-j)/(2j+1)]\langle \mathcal{S}_{1/2}\cdot \mathcal{S}_j\rangle \tag{147}$$

and

$$g^2 = (\overset{\text{stat}}{g})^2 - \frac{2}{(2j+1)j(j+1)}\langle \mathcal{S}_{1/2}\cdot \mathcal{S}_j\rangle^2. \tag{148}$$

The model gives

$$\langle \mathcal{S}_{1/2}\cdot \mathcal{S}_j\rangle = j\left(\frac{j+1}{6}\right)^{1/2}\lambda\lambda_j^1 \boldsymbol{e}\cdot\boldsymbol{e}_j^1, \tag{149}$$

where $\lambda \equiv \lambda^1_{1/2}$ and $\boldsymbol{e} \equiv \boldsymbol{e}^1_{1/2}$. This model expression should be compared with the main formula (8) of the work (Hambro and Mukhopadhyay 1975). We obtain the coincidence provided that

$$\boldsymbol{b} \equiv [\tfrac{2}{3}(j+1)]^{1/2}\lambda^1_j \boldsymbol{e}^1_j \Rightarrow |\boldsymbol{b}| \leqslant [\tfrac{2}{3}(j+1)]^{1/2} \tag{150}$$

contrary to the incorrect claim that $|\boldsymbol{b}| \leqslant 1$. We discussed this mistake after (42a) in § 4. Therefore, the conclusions of Hambro and Mukhopadhyay (1975) are correct only for the case $j = \frac{1}{2}$, which has been considered previously by Bukhvostov and Popov (1964).

In what follows we will need the model expression for the g-function:

$$g = \left\{ \frac{4j+1}{4j+2} - \frac{j}{6j+3}(\lambda\lambda^1_j \boldsymbol{e}\cdot\boldsymbol{e}^1_j)^2 \right\}^{1/2}. \tag{151}$$

The parameters $\langle \mathscr{R}^L_{fe} \rangle_0$ can be calculated from (125)-(126),

$$\begin{aligned}
&\hat{L}\langle \boldsymbol{R}^L_{0L}\rangle_0 = (2j)^{1/2}\lambda^L_j \boldsymbol{e}^L_j \\
&[3(2L-1)]^{1/2}\langle \boldsymbol{R}^L_{1L-1}\rangle_0 = (2j)^{1/2}\lambda\lambda^{L-1}_j(\boldsymbol{e}\otimes\boldsymbol{e}^{L-1}_j)^L \\
&[3(2L+3)]^{1/2}\langle \boldsymbol{R}^L_{1L+1}\rangle_0 = (2j)^{1/2}\lambda\lambda^{L+1}_j(\boldsymbol{e}\otimes\boldsymbol{e}^{L+1}_j)^L.
\end{aligned} \tag{152}$$

(Here one should remember that (37) and (41)$\Rightarrow(2j)^{1/2}\lambda^0_j = 1$.) The most interesting is the case of the cylindrical symmetries of both the initially free subsystems. We note that the situation when the directions of these symmetries (of both subsystems) do *not* coincide has been considered in the unpublished report by Bukhvostov *et al* (1972). We discuss this general case in § 10. Here we like to restrict ourself to the simplest situation of the initial axial symmetry with the *common* direction of the symmetry. Then we can write

$$\begin{aligned}
&\boldsymbol{e}^L_F\cdot(\boldsymbol{e}\otimes\boldsymbol{e}^{L\pm1}_j)^L = \varepsilon^L_F\varepsilon^{L\pm1}_j C^{L0}_{10L\pm10} \\
&\boldsymbol{e}^L_j\cdot\boldsymbol{e}^L_F = \varepsilon^L_j\varepsilon^L_F.
\end{aligned} \tag{153}$$

Inserting (152)-(153) into (108) and recalling the notation (69) we get for $L = 1, 2, 3$

$$\begin{aligned}
&3^{1/2}\hat{j}^3 g\bar{\lambda}^1_+ = j[3(2j+3)]^{1/2}\bar{\lambda}^1_j + [\tfrac{1}{2}(j+1)(2j+3)]^{1/2}\lambda + 2j[\tfrac{1}{5}(2j-1)]^{1/2}\lambda\bar{\lambda}^2_j; \\
&3^{1/2}\hat{j}^3 g\bar{\lambda}^1_- = [3j(j+1)(2j-1)]^{1/2}\bar{\lambda}^1_j - [\tfrac{1}{2}j(2j-1)]^{1/2}\lambda - 2[\tfrac{1}{5}j(j+1)(2j+3)]^{1/2}\lambda\bar{\lambda}^2_j; \\
&5^{1/2}\hat{j}^3 g\bar{\lambda}^2_+ = \{5j(j+2)(2j-1)\}^{1/2}\bar{\lambda}^2_j + 2\{\tfrac{1}{3}j(j+2)(2j+3)\}^{1/2}\lambda\bar{\lambda}^1_j \\
&\qquad + 3\{\tfrac{1}{7}j(j-1)(2j-1)\}^{1/2}\lambda\bar{\lambda}^3_j; \\
&5^{1/2}\hat{j}^3 g\bar{\lambda}^2_- = \{5j(j-1)(2j+3)\}^{1/2}\bar{\lambda}^2_j - 2\{\tfrac{1}{3}j(j-1)(2j-1)\}^{1/2}\lambda\bar{\lambda}^1_j \\
&\qquad - 3\{\tfrac{1}{7}j(j+2)(2j+3)\}^{1/2}\lambda\bar{\lambda}^3_j; \\
&7^{1/2}\hat{j}^3 g\bar{\lambda}^3_+ = \{7j(j-1)(2j+5)\}^{1/2}\bar{\lambda}^3_j + 3\{\tfrac{1}{5}j(j+2)(2j+5)\}^{1/2}\lambda\bar{\lambda}^2_j \\
&\qquad + \tfrac{4}{3}\{j(j-1)(2j-3)\}^{1/2}\lambda\bar{\lambda}^4_j; \\
&7^{1/2}\hat{j}^3 g\bar{\lambda}^3_- = \{7j(j+2)(2j-3)\}^{1/2}\bar{\lambda}^3_j - 3\{\tfrac{1}{5}j(j-1)(2j-3)\}^{1/2}\lambda\bar{\lambda}^2_j \\
&\qquad - \tfrac{4}{3}\{j(j+2)(2j+5)\}^{1/2}\lambda\bar{\lambda}^4_j.
\end{aligned} \tag{154}$$

These formula cover the cases $\rho(\frac{1}{2}\otimes\frac{1}{2})$ and $\rho(\frac{1}{2}\otimes 1)$. The comparison with equations (12) in Bukhvostov *et al* (1972) show the misprint in the formula for λ^{BP}_-. It should be stressed that the g-function in the formulae (154) is given by the expression (151).

Formulae (154) follow from the model of the depolarisation and can be tested experimentally.

For the case $\rho(\frac{1}{2}\otimes\frac{1}{2})$ we have finally $(j=\frac{1}{2})$

$$\begin{aligned} 2(3)^{1/2}g &= [9-(\lambda\lambda_j^1)^2]^{1/2} \\ 2(2)^{1/2}g\bar{\lambda}_+^1 &= \lambda + \bar{\lambda}_j^1 \\ 6^{1/2}g\bar{\lambda}_+^2 &= \lambda\bar{\lambda}_j^1 \\ p_+ = \tfrac{3}{4}+\tfrac{1}{4}\lambda\bar{\lambda}_j^1, &\qquad \bar{\lambda}_j^1 \equiv \lambda_j^1 \boldsymbol{e}\cdot\boldsymbol{e}_j^1. \end{aligned} \tag{155}$$

Here and in (154) λ is non-negative, $(\boldsymbol{\sigma}\equiv\boldsymbol{e})$,

$$2\boldsymbol{\sigma}\cdot\langle\mathcal{S}_{1/2}\rangle = \lambda = f \geqslant 0 \quad (\text{cf } (80)). \tag{156}$$

Inserting (41) into (78),

$$3^{1/2}\boldsymbol{s}\cdot\langle\mathcal{S}_j\rangle = j[2(j+1)]^{1/2}\bar{\lambda}_j^1 = 3^{1/2}jf_j^1. \tag{157}$$

The degrees of the orientation of the whole atom $\bar{\lambda}_F^L$ should be read off from (68), (70), (78). One can use also the identities (112)-(114),

$$\begin{aligned} \langle\mathcal{P}_+^1(m)\circ P_+\rangle &= 2^{1/2}g\bar{\lambda}_+^1 = f_+^1 \\ \langle\mathcal{P}_+^2(m)\circ P_+\rangle &= \tfrac{2}{3}^{1/2}g\bar{\lambda}_+^2 = f_+^2. \end{aligned} \tag{158}$$

A more realistic model for the system $\rho(\frac{1}{2}\otimes\frac{1}{2})$ has been considered by Bukhvostov and Popov (1964). Our only innovation in (155)-(158) is that our degrees of the orientation are truly independent of the populations i.e. that $0\leqslant\lambda_F^L\leqslant 1$. As a consequence the g-function occurs which makes the dependence of the atom polarisation on the initial polarisations correct.

The case $\rho(\frac{1}{2}\otimes 1)$ has been considered by Bukhvostov *et al* (1972) adopting the factorised form of the spin-density operator. Now from (154) we obtain

$$\begin{aligned} 9(2)^{1/2}g\bar{\lambda}_-^1 &= -\lambda + 2(3)^{1/2}\bar{\lambda}^1 - 4\lambda\bar{\lambda}^2 \\ 9(5)^{1/2}g\bar{\lambda}_+^1 &= 5\lambda + 5(3)^{1/2}\bar{\lambda}^1 + 2\lambda\bar{\lambda}^2 \\ 3(3)^{1/2}g\bar{\lambda}_+^2 &= 2\lambda\bar{\lambda}^1 + 3^{1/2}\bar{\lambda}^2 \\ 5^{1/2}g\bar{\lambda}_+^3 &= \lambda\bar{\lambda}^2 \end{aligned} \tag{159}$$

where $3(2)^{1/2}g=[15-2(\lambda\lambda^1)^2]^{1/2}$, and $p_+=\frac{2}{3}+\frac{2}{9}3^{1/2}\lambda\bar{\lambda}^1$. Formulae (108), (154) for the case $\rho(\frac{1}{2}\otimes j)$ imply the $2j$ model-dependent relations (rather complicated, as one can see from (159)) which can be tested experimentally.

10. SO(3) Clebsch–Gordan coefficients for the different bases

The evaluation of the SO(3) Clebsch-Gordan coefficients for the different bases in the space of the representation of SO(3) group allows us to enlarge essentially the results of § 9 to the case when the directions of the cylindrical symmetrics of the initially free subsystems do not coincide. We would like to give here the self-contained discussion of this subject which also plays a crucial role in the study of the angular and polarisation distributions in the particle reactions and decays. Our approach is based on the notion of the tensor operator (§§ 2 and 3) and can be applied also to the other Lie groups.

The novelty of our presentation is due to the explicit dependence of the SO(3) irreducible bases in $[j]$ on the basis in $\mathscr{G} \simeq \boldsymbol{R}^3$ and in particularly on the axis s of the spin quantisation, i.e. we are diagonalising the coordinate independent Hermitian operator $\mathscr{F}(s)$, cf Steiger and Fritz (1967). The different bases in $[j]$ correspond to the different bases in the SO(3) Lie algebra.

To each ordered and orthonormal (WRT $\phi(14)$) basis e in $\mathscr{G}$, i.e. to the ordered triple of the unit mutually orthogonal vectors, $e \equiv \{e_1, e_2, e_3\}$, one can associate the Cartan basis (with frame) in the complexified SO(3) Lie algebra and, in this way, specify the basis in $[j]: e \to \{|j\mu e\rangle\}$. The different bases in $[j]$ correspond to the diagonalisation of the different elements in the Lie algebra, i.e. to the different axes of the quantisation, and are related to the different choices of the phases in the complex one-dimensional eigensubspaces.

The aim of this section is the evaluation of the SO(3) Clebsch-Gordan coefficient

$$C^{D\delta}_{A\alpha B\beta}(abd) = \langle (AB)D\delta | \{ |A\alpha a\rangle \otimes |B\beta b\rangle \}, \tag{160}$$

for the three different $\mathscr{G}$ bases corresponding to three different quantisation axes. If the $\mathscr{G}$ bases coincide, $a = b = d$, then (160) is reduced to the usual basis independent Clebsch-Gordan coefficient (Varschalovich *et al* 1975, Jucys and Bandzaitis 1977), $C^{D\delta}_{A\alpha B\beta} \in \boldsymbol{R}$.

Let us see how the Wigner $\mathscr{D}$-functions, which are the matrix elements of the irreducible representation $G \ni g \mapsto \mathscr{D}(g) \in \operatorname{End} \mathscr{H}$ are related to the tensor operators (6). The G-invariance of the mapping (6), $\mathscr{T}: \mathscr{L} \to \operatorname{End} \mathscr{H}$ means that (Werle 1966, formula (8.13)),

$$\mathscr{D}(g) \circ \mathscr{T} = \mathscr{T}g \circ \mathscr{D}(g) : \mathscr{L} \to \operatorname{End} \mathscr{H}. \tag{161}$$

One should note that the above equation is not compatible with Klimyk's definition, (cf equations (2)-(3) in Klimyk 1983). As a consequence of (161) we have for the SO(3) group the coordinate free relations

$$[\mathscr{F}(n), \mathscr{F}(m)] = \mathrm{i}\mathscr{F}(n \times m)$$

where $n \times m \equiv \mathscr{F}(n)m$, and

$$\mathscr{D}(g)|j\mu e\rangle = |j\mu g e\rangle. \tag{162}$$

The Wigner $\mathscr{D}$ function is defined as

$$\mathscr{D}^j_{\alpha\beta}(a \to b) \equiv \langle j\alpha a | j\beta b\rangle. \tag{163}$$

This gives the main formula of this section,

$$C^{D\delta}_{A\alpha B\beta}(abd) = \mathscr{D}^D_{\delta\rho}(d \to e) C^{D\rho}_{A\mu B\nu}(e) \mathscr{D}^A_{\mu\alpha}(e \to a) D^B_{\nu\beta}(e \to b) \tag{164}$$

which is independent of the basis e. The well known symmetry properties of the usual basis-independent Clebsch-Gordan coefficients imply the corresponding symmetries for different bases. For example from (164) we get

$$C^{D}_{A\alpha B\beta}(abd) = (-)^{A+B-D} C^{D}_{B\beta A\alpha}(bad).$$

Next we would like to evaluate the explicit expressions for the important Clebsch-Gordan coefficients $C^{D0}_{A0B0}(abd)$, usually referred to as the invariant angular functions because they can be expressed in terms of the scalar products of the corresponding vectors. The explicit expressions and the recurrent relations for some of the $C^{D0}_{A0B0}(abd)$ Clebsch-Gordan coefficients have been presented by MacFarlane (1962). The Clebsch-Gordan coefficient $C^{D0}_{A0B0}(abd)$ coincides with the invariant angular function $S_{ABD}(abd)$ from Ciechanowicz and Oziewicz (1984, formula (3.26)). These particular Clebsch-Gordan coefficients in fact do not depend on the full bases a, b and d in $\mathscr{G}$; they are completely determined, as we will see, by the unit vectors, denoted here by $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{d}\}$, describing the corresponding axes of the spin quantisations.

Accepting the parametrisation of the rotations through the Euler angles according to the convention from the monograph by Werle (1966, § 11) we have the following relation of the Wigner functions to the Condon and Snortley (1935) spherical functions (formula (12.18) from Werle 1966)

$$\hat{l}d^l_{m0} = (4\pi)^{1/2}\overset{*}{Y}{}^l_m,$$

where

$$(4\pi)^{1/2}Y^l_m = \hat{l}\,\mathrm{e}^{im\varphi}\left\{\frac{(l-|m|)!}{(l+|m|)!}\right\}^{1/2}(-\mathrm{sign}\, m\cdot\sin\Theta)^{|m|}\mathrm{P}^{(|m|)}_l. \tag{165}$$

Inserting (165) into (164) we get three different expressions for the Clebsch-Gordan coefficients $C^{D0}_{A0B0}(abd)$ by taking $e=d$, a and b in (164) correspondingly. In these expressions it will be convenient to introduce a shorthand abbreviation for the scalar complex angular function:

$$[\boldsymbol{abd}] \equiv (\boldsymbol{a}\times\boldsymbol{d})\cdot(\boldsymbol{b}\times\boldsymbol{d}) + \mathrm{i}(\boldsymbol{a}\times\boldsymbol{b})\cdot d \tag{166}$$

where one can use $(\boldsymbol{a}\times\boldsymbol{d})\cdot(\boldsymbol{b}\times\boldsymbol{d}) = \boldsymbol{a}\cdot\boldsymbol{b} - (\boldsymbol{a}\cdot\boldsymbol{d})(\boldsymbol{b}\cdot\boldsymbol{d})$. Below in formulae (167) we will also denote

$$[\ldots]^k_\pi \equiv \begin{cases} \mathrm{Re}[\ldots]^k & \text{for } \pi = \text{even} \\ \mathrm{i}\ \mathrm{Im}[\ldots]^k & \text{for } \pi = \text{odd}. \end{cases}$$

$$\begin{aligned} C^{D0}_{A0B0}(abd) &= C^{D0}_{A0B0}P_A(\boldsymbol{a}\cdot\boldsymbol{d})P_B(\boldsymbol{b}\cdot\boldsymbol{d}) \\ &\quad + 2\sum_{k>0}(-)^k C^{D0}_{AkB-k}\left\{\frac{(A-k)!(B-k)!}{(A+k)!(B+k)!}\right\}^{1/2} \\ &\quad \times \mathrm{P}^{(k)}_A(\boldsymbol{a}\cdot\boldsymbol{d})\mathrm{P}^{(k)}_B(\boldsymbol{b}\cdot\boldsymbol{d})[\boldsymbol{abd}]^k_{A+B-D}; \end{aligned} \tag{167a}$$

$$\begin{aligned} C^{D0}_{A0B0}(abd) &= C^{D0}_{A0B0}\mathrm{P}_B(\boldsymbol{a}\cdot\boldsymbol{d})\mathrm{P}_D(\boldsymbol{a}\cdot\boldsymbol{d}) \\ &\quad + 2\sum_{k>0} C^{Dk}_{A0Bk}\left\{\frac{(B-k)!(D-k)!}{(B+k)!(D+k)!}\right\}^{1/2} \\ &\quad \times \mathrm{P}^{(k)}_B(\boldsymbol{a}\cdot\boldsymbol{b})\mathrm{P}^{(k)}_D(\boldsymbol{a}\cdot\boldsymbol{d})[\boldsymbol{bda}]^k_{A+B-D}; \end{aligned} \tag{167b}$$

$$\begin{aligned} C^{D0}_{A0B0}(abd) &= C^{D0}_{A0B0}\mathrm{P}_A(\boldsymbol{a}\cdot\boldsymbol{b})\mathrm{P}_D(\boldsymbol{b}\cdot\boldsymbol{d}) \\ &\quad + 2\sum_{k>0} C^{Dk}_{AkB0}\left\{\frac{(A-k)!(D-k)!}{(A+k)!(D+k)!}\right\}^{1/2} \\ &\quad \times \mathrm{P}^{(k)}_A(\boldsymbol{a}\cdot\boldsymbol{b})\mathrm{P}^{(k)}_D(\boldsymbol{b}\cdot\boldsymbol{d})[\boldsymbol{abd}]^k_{A+B-D}. \end{aligned} \tag{167c}$$

As an example let us consider the particular case of the above formulae which one needs most often in the applications: the case $A=1$ for which the most convenient are the expressions (167a) and (167c) giving at most two terms,

$$C^{s0}_{10s0}(abd) = -\mathrm{i}[s(s+1)]^{-1/2}\boldsymbol{a}\times\boldsymbol{b}\cdot\boldsymbol{d}\,P'_s(\boldsymbol{b}\cdot\boldsymbol{d}),$$
$$C^{s0}_{10s-10}(abd) = [s(2s-1)]^{-1/2}\{\boldsymbol{a}\cdot\boldsymbol{d}\,P'_s(\boldsymbol{b}\cdot\boldsymbol{d}) - \boldsymbol{a}\cdot\boldsymbol{b}\,P'_{s-1}(\boldsymbol{b}\cdot\boldsymbol{d})\}, \tag{168}$$
$$C^{s0}_{10s+10}(abd) = [(s+1)(2s+3)]^{-1/2}\{\boldsymbol{a}\cdot\boldsymbol{d}\,P'_s(\boldsymbol{b}\cdot\boldsymbol{d}) - \boldsymbol{a}\cdot\boldsymbol{d}\,P'_{s+1}(\boldsymbol{b}\cdot\boldsymbol{d})\}.$$

Here we have used the recurrent relations $P'_s = xP'_{s+1} - (s+1)P_{s+1} = sP_{s-1} + xP'_{s-1}$.

The states $|j0s\rangle \in [j]$, deserve the special attention: we will show that $|j0s\rangle = \boldsymbol{s}^j$, i.e. that these states coincide with the rank-j tensors introduced in (65). What is special about these states is that they in fact do not depend on the full basis s in $\mathscr{G}$: they are completely determined by the unit vector $\boldsymbol{s}$, describing the axis of the spin quantisation. Rotations around the quantisation axis $\boldsymbol{s}$ leave the states $\boldsymbol{s}^j = |j0\boldsymbol{s}\rangle$ unchanged, as it follows from (162). These states define the mapping $\boldsymbol{s}\to\boldsymbol{s}^j$, i.e. the embedding (65) of the real spheres $\mathscr{S}^2 \to \mathscr{S}^{2j}$. The explicit evaluation of this embedding will show the equality $\boldsymbol{s}^j = |j0s\rangle$. In fact the embedding (65) is induced by the mapping

$$\mathscr{G} \ni \boldsymbol{s} \mapsto \mathrm{Ker}\,\mathscr{F}(\boldsymbol{s}) \subset \mathscr{H}, \tag{169}$$

where $\dim \mathrm{Ker}\,\mathscr{F}(\boldsymbol{s}) = 1$ for $\forall \boldsymbol{s}\in\mathscr{G}$ and for $\mathscr{H} = [j]$, $j =$ integer. We have $\mathscr{F}(\boldsymbol{s})|j0s\rangle = 0$ and $|j0s\rangle\cdot|j0s\rangle = 1$, (15). For the spherical components s^j_λ of the vector $s^j = |j0s\rangle$ we get

$$|j0s\rangle = |j\lambda r\rangle\langle j\lambda r|j0s\rangle$$
$$= |j\lambda r\rangle \mathscr{D}_{\lambda 0}(r\to s). \tag{170}$$

Therefore

$$s^j_\lambda(r) = \mathscr{D}^j_{\lambda 0}(r\to s) \Rightarrow \overset{*}{s}{}^j_\lambda = (-)^\lambda s^j_{-\lambda}, \tag{171}$$

see formula (12.48) in Werle (1966). The vectors with the property (171) are referred to as the *real* vectors. For real vectors $\boldsymbol{a}\cdot\boldsymbol{b} = \Sigma\,(-)^\lambda a_\lambda b_{-\lambda} \in \boldsymbol{R}$. Therefore $|j0s\rangle \in \mathscr{S}^{2j}$ as it follows from (171).

Because

$$(\boldsymbol{a}^A \otimes \boldsymbol{b}^B)^D \equiv \mathrm{P}^D_{AB}\cdot\{|A0a\rangle\otimes|B0b\rangle\}, \tag{172}$$

where P^D_{AB} is the projector on the SO(3) irreducible subspace $[D]\subset[A]\otimes[B]$, then using (160), we have

$$(\boldsymbol{a}^A \otimes \boldsymbol{b}^B)^D = \sum |D\delta d\rangle C^{D\delta}_{A0B0}(abd). \tag{173}$$

The above formula for $a = b = d = s$ coincide with the recurrent relation (65) which show explicitly that $\boldsymbol{s}^j = |j0s\rangle$ and moreover the way in which $\boldsymbol{s}^j$ is built up from $\boldsymbol{s}\in\mathscr{G}$.

The following formulae are some of the simple consequences of the above discussion,

$$\boldsymbol{a}^L\cdot\boldsymbol{b}^L = \langle L0\boldsymbol{a}|L0\boldsymbol{b}\rangle = d^L_{00}(\boldsymbol{a}\to\boldsymbol{b}) = \mathrm{P}_L(\boldsymbol{a}\cdot\boldsymbol{b}). \tag{174}$$

$$(\boldsymbol{a}^A\otimes\boldsymbol{b}^B)^D\cdot\boldsymbol{d}^D = C^{D0}_{A0B0}(abd). \tag{175}$$

$$(\boldsymbol{a}^A\otimes\boldsymbol{b}^B)^E\cdot(\boldsymbol{a}^C\otimes\boldsymbol{b}^D)^E$$
$$= (-)^{C+B}\hat{E}^2\sum C^{H0}_{A0C0}C^{H0}_{B0D0}\begin{Bmatrix} A & B & E\\ D & C & H\end{Bmatrix}\mathrm{P}_H(\boldsymbol{a}\cdot\boldsymbol{b}). \tag{176}$$

These formulae are indispensable for the calculation of the angular and polarisation distributions in the scattering and decay processes.

Now we can come back to the main subject of this paper, which is the calculation of the degrees of the orientation of the atom $\{\lambda^L_F\}$, according to (94), in the model of the depolarisation due to the hyperfine spin-spin interaction i.e. with the help of (125)-(126). We wish to generalise the results of § 9 related to the case of the cylindrical symmetries of the both initially free subsystems. Then according to (67) we can put in (125)-(126)

$$\boldsymbol{e}^f_{j_1} = \varepsilon^f_{j_1}\boldsymbol{r}^f \qquad \text{and} \qquad \boldsymbol{e}^e_{j_2} = \varepsilon^e_{j_2}\boldsymbol{s}^e.$$

Therefore the problem of the evaluation of the $\langle \boldsymbol{R}^L_{fe}\rangle_0$ parameters (125) is reduced to the calculation of the $(\boldsymbol{r}^f \otimes \boldsymbol{s}^e)^L$ angular function according to (173) above, i.e. we must calculate another kind of Clebsch-Gordan coefficient for the different bases, namely $C^{D\delta}_{A0B0}(abd)$ with $d = a$ or b. From (164) we have

$$C^{D\delta}_{A0B0}(aba) = C^{D\delta}_{A0B0}\mathscr{D}^B_{\delta 0}(a \to b),$$

and

$$C^{D\delta}_{A0B0}(abb) = C^{D\delta}_{A\delta B0}\mathscr{D}^A_{\delta 0}(b \to a). \tag{177}$$

The above formulae are sufficient for the model calculations of the atom degrees of the orientation for any particular two-spin systems. For instance (176) jointly with (94) allows us to calculate the square of the non-negative degrees of the orientation of the atom $(\lambda^L_F)^2$, in terms of the initial degrees of the orientation $\{\bar{\lambda}^f_{j_1}, \bar{\lambda}^e_{j_2}\}$ and in terms of $\mathrm{P}_H(\boldsymbol{r}\cdot\boldsymbol{s})$, generalising in this way (154).

10. Conclusion

We have presented the brief theory of the tensor and polarisation operators in the basis free manner. Although we were interested in the SO(3) irreducible tensor operators our presentation could be generalised and used for arbitrary Lie groups. Then we apply this formalism to the multipole expansion of the spin-density operator. The multipole expansion (38) is, strictly speaking, valid only when the space $\mathscr{H}$ does not contain the multiple irreducible representations of the SO(3) group. The generalisation to the case which takes into account accurately the multiplicities of the irreducible decomposition of $\mathscr{H}$ can be done easily (see Klimyk (1983)).

The most essential conclusion is that the spin-density operator can *not* be presented in the factorised form (4) as we have discussed already in the introduction. The multipole expansion (38) gives, among other things the correct model-dependent expressions of § 9.

We have considered in some detail the spin-density operator for the interacting *two* arbitrary spin systems in §§ 5-7. These results should help with the phenomenological analysis of the decays of the muonic atoms and other nuclear reactions. For example, they will be useful for the investigation of the nuclear muon-capture reaction by oriented or polarised targets with *arbitrary* spin. We hope that some of our results can find applications also in the complicated theory of the atomic depolarisation of muons as developed by Djrbashyan (1959), Shmushkevich (1959) and mostly by Bukhvostov (1966, 1969).

The last sections are devoted to the simplest model of the depolarisation. The paper does not exhaust the subject: more reaslistic models of the formation of the atom need to be investigated along the lines developed by Bukhvostov.

Appendix

We list here some formulae for the SO(3) Fano, Racah and Clebsch-Gordan coefficients which we used frequently in this paper. They are taken from the excellent monographs by Varschalovich *et al* (1975) and by Jucys and Bandzaitis (1977).

$$2^{1/2}\hat{f}\hat{L}(\hat{j}\hat{F})^2 \begin{Bmatrix} \frac{1}{2} & j & F \\ \frac{1}{2} & j & F \\ f & e & L \end{Bmatrix} = C^{L0}_{f0e0} W^F_j(e, L),$$

where

$$W^F_j(e, L) \equiv \begin{cases} \left(\dfrac{(2j+L+2)!(2j+1-L)!}{(2j+e+1)!(2j-e)!}\right)^{1/2} & \text{for } F = j+\frac{1}{2} \\ (-)^{L-e}\left(\dfrac{(2j+e+1)!(2j-e)!}{(2j+L)!(2j-L-1)!}\right)^{1/2} & \text{for } F = j-\frac{1}{2}. \end{cases}$$

$$\begin{Bmatrix} j_2 & F & j_1 \\ 1 & j_1 & F \end{Bmatrix} = (-)^{1+F+j_1+j_2} \frac{F(F+1)+j_1(j_1+1)-j_2(j_2+1)}{2\hat{F}\hat{j}_1[j_1(j_1+1)F(F+1)]^{1/2}}.$$

$$C^L_{10L+10} = -\left(\frac{L+1}{2L+3}\right)^{1/2}, \qquad C^L_{10L-10} = +\left(\frac{L}{2L-1}\right)^{1/2}.$$

Acknowledgments

I wish to dedicate the paper to my parents.

I am greatly indebted to Dr N P Popov from the Leningrad Institute of Nuclear Physics (Gatchina, USSR) for stimulating and valuable correspondence over the last 15 years.

The present paper could never have been completed without the considerable help of my wife Christina and my friend Jerzy Kocik.

I would like to thank the procecutor mgr Andrzej Kaucz who gave me permission to perform all this work in the Wrocław Jail of the Home Office and in the Wrocław Jail of Inquiry in the period March-July 1984.

References

Bukhvostov A P 1966 *Yad. Fiz.* **4** 83

—— 1969 *Yad. Fiz.* **9** 107

Bukhvostov A P, Oziewicz Z and Popov N P 1972 *Muon capture by oriented spin one nuclei* (Leningrad Nuclear Physics Institute: Gatchina) *unpublished report No 324*

Bukhvostov A P and Popov N P 1964 *Ž. Eksp. Teor. Fiz.* **46** 1842 (Engl. trans. *Sov. Phys.-JETP* **19** 1240)

—— 1967 *Yad. Fiz.* **6** 809

—— 1970 *Nucl. Phys.* A **147** 385

Bukhvostov A P, Vartanyan V A, Oziewicz Z, Popov N P and Eramzhyan R A 1971 *Yad. Fiz.* **13** 820

Ciechanowicz S and Mukhopadhyay N C 1978 *On the average recoil polarization in the nuclear muon capture (JINR Dubna) E4-11888*

Ciehanowicz S and Oziewicz Z 1984 *Fortschr. Phys.* **32** 61–88

Condon E U and Shortley G H 1935 *The theory of Atomic Spectra* (Cambridge: CUP)

Csonka P L, Moravcsik M J and Scadron M D 1966 *Ann. Phys., NY* **40** 100

Devons S and Goldfarb L J B 1957 in *Handbuch der Physik* vol 42 (Berlin: Springer)
Djrbashyan V A 1959 *Ž. Eksp. Teor. Fiz.* **36** 277
Edmonds A R 1957 *Angular Momentum in Quantum Mechanics* (Princeton: Princeton University Press)
Fano U 1951 *Report No 1214* (Washington: National Bureau of Standards)
Fano U and Racah G 1959 *Irreducible Tensorial Sets* (New York: Academic)
Goldfarb L J B and Bromley D A 1962 *Nucl. Phys.* **39** 408
Hambro H and Mukhopadhyay N C 1975 *Lett. Nuovo Cimento* **14** 53
Jucys A P and Bandzaitis A A 1977 *Theory of Angular Momentum in Quantum Mechanics* 2nd edn (in Russian) (Vilnius: Mokslas)
Klimyk A U 1983 *J. Phys. A: Math. Gen.* **16** 3693
MacFarlane A J 1962 *Nucl. Phys.* **38** 504
Mukhopadhyay N C 1977 *Phys. Rep.* **30C** 1
Oziewicz Z 1977 *Acta Phys. Polon.* B **8** 563
Rose M E 1957 *Elementary Theory of Angular Momentum* (New York: Wiley)
—— 1962 *Polarization phenomena in beta and gamma emission* in *Lectures in Theoretical Physics, Brandeis Summer Institute* (New York: Benjamin)
Schiff L I 1968 *Quantum Mechanics* (New York: Wiley)
Shmushkevich I M 1959a *Ž. Eksp. Teor. Fiz.* **36** 953
—— 1959b *Nucl. Phys.* **11** 419
Steiger A D and Fritz K J 1967 *Nuovo Cimento* **51A** 461
Tolhoek H A and Cox J A M 1953 *Physica* **19** 101
Varschalovich D V, Moskalev A N and Khersonsky V K 1975 *Quantum Theory of Angular Momentum* (in Russian) (Leningrad: Nauka)
Werle J 1966 *Relativistic Theory of Reactions* (Amsterdam: North-Holland)